1 Hamiltonian mechanics

> Just now I am teaching the foundations of poor deceased mechanics, which is so beautiful. What will her successor look like? With that question I torment myself incessantly.
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> ALBERT EINSTEIN, letter to Heinrich Zangger, 14 November 1911

1.1. Hamilton's equations

Note. We shall use Einstein's summation convention that, unless otherwise indicated, expressions are summed over all repeated indices.

Around the end of the eighteenth century Joseph-Louis Lagrange discovered that Newton's laws of mechanics for a system with n degrees of freedom could be very succinctly expressed in terms of a single function L. Writing t for time, $x_1, \ldots, x_n$ for the position coordinates, $v_1 = \dot{x}_1, \ldots, v_n = \dot{x}_n$ for their velocities, T for the kinetic energy and V for the potential energy, one defines

$$L(x_1, \ldots, x_n, v_1, \ldots, v_n, t) = T - V.$$

Lagrange's equations for the motion of a holonomic system with conservative forces can be written as

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v_j}\right] - \frac{\partial L}{\partial x_j} = 0,$$

where the x_j's are the generalized coordinates, $v_j = \dot{x}_j$ the components of generalized velocity, and L is the Lagrangian.

Lagrange and Sir William Rowan Hamilton noticed that the equations of motion are precisely the Euler–Lagrange equations for finding the minimum of the integral

$$\int L(x_1(t), \ldots, x_n(t), v_1(t), \ldots, , v_n(t), t)\, dt,$$

a fact now known as *Hamilton's Principle.* Lagrange's equations (like Newton's) are second order differential equations, but Hamilton also noticed that they could be replaced by first order equations in twice as many variables by adopting the following definitions.

Definition 1.1.1. The expression $p_j = \partial L / \partial v_j$ is called the j-th component of *generalized momentum.*

Example 1.1.1. A particle of mass m moving in a potential $V(\mathbf{x})$.

Since the Lagrangian is given by

$$L = T - V = \frac{1}{2} m \left(v_1^2 + v_2^2 + v_3^2\right) - V(\mathbf{x}),$$

we obtain

$$p_j = \frac{\partial L}{\partial v_j} = m v_j,$$

that is the usual linear momentum.

Example 1.1.2. Motion in a plane under the influence of a central potential.
In polar coordinates the Lagrangian is

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - V(r),$$

so that the generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad \text{and} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}.$$

We see immediately that p_θ is the angular momentum of the particle, which provides additional support for the name generalized momentum.

Definition 1.1.2. The *Hamiltonian*, h, is defined by

$$h = p_j x_j - L.$$

The Hamiltonian is apparently a function of the x's, the v's, the p's and t, but usually one can solve the equation defining p_j to find the velocities in terms of the momenta and so write h as a function just of coordinates, momenta and time.

The derivatives of the Hamiltonian can either be found directly, using the chain rule, or by the following argument. Small changes δx_j in x_j, δp_j in p_j, and δt in t lead to small changes δv_j in the velocities and δh in the Hamiltonian. In fact, by definition we have

$$\delta h = v_j \delta p_j + p_j \delta v_j - \frac{\partial L}{\partial v_j}\delta v_j - \frac{\partial L}{\partial x_j}\delta x_j - \frac{\partial L}{\partial t}\delta t + o(\delta x, \delta p, \delta t),$$

Where the last term is the remainder which tends to zero faster than any of its arguments. The second and third terms cancel by virtue of the definition of momentum, leaving

$$\delta h = v_j \delta p_j - \frac{\partial L}{\partial x_j}\delta x_j - \frac{\partial L}{\partial t}\delta t + o(\delta x, \delta p, \delta t).$$

From this we can read off the derivatives of h as

$$\frac{\partial h}{\partial p_j} = v_j, \qquad \frac{\partial h}{\partial x_j} = -\frac{\partial L}{\partial x_j}, \qquad \frac{\partial h}{\partial t} = -\frac{\partial L}{\partial t}.$$

Finally, we recall that

$$\frac{dp_j}{dt} = \frac{d}{dt}\left[\frac{\partial L}{\partial v_j}\right] = \frac{\partial L}{\partial x_j} = -\frac{\partial h}{\partial x_j},$$

and

$$\frac{dx_j}{dt} = v_j = \frac{\partial h}{\partial p_j}.$$

Definition 1.1.3. The equations

$$\frac{dx_j}{dt} = \frac{\partial h}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial h}{\partial x_j}$$

are known as *Hamilton's equations.*

The Hamiltonian for the particle in three dimensions is given by

$$\begin{aligned} h &= p_j v_j - L \\ &= \frac{1}{m}\left(p_1^2 + p_2^2 + p_3^2\right) - \frac{1}{2}m\left[\left(\frac{p_1}{m}\right)^2 + \left(\frac{p_2}{m}\right)^2 + \left(\frac{p_3}{m}\right)^2\right] + V \\ &= \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + V. \end{aligned}$$

For the equations of motion we obtain

$$\begin{aligned} \frac{dx_j}{dt} &= \frac{\partial h}{\partial p_j} &&= \frac{1}{m}p_j, \\ \frac{dp_j}{dt} &= -\frac{\partial h}{\partial x_j} &&= -\frac{\partial V}{\partial x_j}, \end{aligned}$$

the first of which is equivalent to the definition of the momentum in terms of the velocity. The second equation is clearly equivalent to the usual Newtonian equation of motion.

Example 1.1.3. A free particle.

When $V = 0$ the second equation reduces to

$$\frac{dp_j}{dt} = 0,$$

which can immediately be integrated to tell us that p_j is a constant. The other equation

$$\frac{dx_j}{dt} = \frac{1}{m}p_j$$

then integrates to

$$x_j(t) = x_j(0) + \frac{1}{m}p_j t.$$

Hamilton's method is particularly good at highlighting constants of the motion such as the momentum here.

Example 1.1.4. The harmonic oscillator.

Let us consider a one-dimensional oscillator with potential $V = \frac{1}{2}m\omega^2 x^2$. As in three dimensions the Hamiltonian is

$$h = \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

and the equations of motion are

$$\frac{dx}{dt} = \frac{1}{m}p, \qquad \text{and} \qquad \frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m\omega^2 x.$$

The solution can be simplified considerably by noticing that the complex variable $p + im\omega x$ satisfies

$$\frac{d}{dt}\left(p + im\omega x\right) = -m\omega^2 x + i\omega p = i\omega\left(p + im\omega x\right).$$

This can be integrated immediately to give

$$p + im\omega x = e^{i\omega t}\left(p_0 + im\omega x_0\right),$$

where p_0 and x_0 are the initial values of p and x. Taking the imaginary part we obtain

$$m\omega x = m\omega x_0 \cos(\omega t) + p_0 \sin(\omega t),$$

which provides the solution of the equations of motion with but a single integration.

Example 1.1.5. Motion in a plane under the influence of a central potential.
The Hamiltonian for central motion in a plane is

$$\begin{aligned} h &= p_r \dot{r} + p_\theta \dot{\theta} - L \\ &= \frac{1}{m}p_r^2 + \frac{1}{mr^2}p_\theta^2 - \frac{1}{2}m\left[\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2\right] + V(r) \\ &= \frac{1}{2m}\left[p_r^2 + \left(\frac{p_\theta}{r}\right)^2\right] + V(r). \end{aligned}$$

This yields the equations of motion

$$\begin{aligned} \frac{dr}{dt} &= \frac{1}{m}p_r, \qquad & \frac{d\theta}{dt} &= \frac{1}{mr^2}p_\theta \\ \frac{dp_r}{dt} &= -\frac{dV}{dr}, \qquad & \frac{dp_\theta}{dt} &= 0. \end{aligned}$$

The last equation immediately gives the conservation of angular momentum: p_θ is constant.
In fact, whenever the Hamiltonian is independent of a coordinate x_j one has

$$\frac{dp_j}{dt} = -\frac{\partial h}{\partial x_j} = 0,$$

so that p_j is a constant.

1.2. The Hamiltonian

The method of constructing a Hamiltonian from a Lagrangian is rather laborious and one would prefer a more direct approach. In fact, if one considers the examples given so far the Hamiltonian has always turned out to be the sum of the kinetic and potential energies, that is the total energy of the system.

Although this is *not* true in general, there is a simple modification which works for most practical problems, where the kinetic energy is a quadratic function in each of the generalized velocities.

Proposition 1.2.1. Suppose that the kinetic energy T can be written as $T = T_2 + T_1 + T_0$ where T_α is homogeneous of degree α in the generalized velocities. Then the Hamiltonian is given by

$$h = T_2 - T_0 + V.$$

Proof. Using Euler's Theorem on homogeneous functions we have

$$p_j v_j = v_j \frac{\partial L}{\partial v_j} = v_j \frac{\partial T}{\partial v_j} = 2 \times T_2 + 1 \times T_1 + 0 \times T_0,$$

so that

$$\begin{aligned} h &= p_j v_j - L = (2T_2 + T_1) - (T_2 + T_1 + T_0 - V) \\ &= T_2 - T_0 + V. \end{aligned}$$

When T contains only second order terms in the velocities we have $T = T_2$, and $h = T_2 + V = T + V$ is just the energy. This is what happened in our previous examples. ♢

1.3. Poisson brackets

From Hamilton's equations we can easily calculate the rate of change of any function f of the p's, x's and t:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial p_j}\frac{dp_j}{dt} + \frac{\partial f}{\partial x_j}\frac{dx_j}{dt} + \frac{\partial f}{\partial t} \\ &= -\frac{\partial f}{\partial p_j}\frac{\partial h}{\partial x_j} + \frac{\partial f}{\partial x_j}\frac{\partial h}{\partial p_j} + \frac{\partial f}{\partial t}. \end{aligned}$$

Definition 1.3.1. The quantity

$$\{g, f\} = \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j}.$$

is known as the *Poisson bracket* of the functions g and f.

We may now write

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{h, f\}.$$

The main properties of the Poisson bracket can be summarized as follows.

Proposition 1.3.1. For all functions f, g and k of the generalized coordinates, momenta and time the Poisson bracket satisfies:
(i) $\{f, g\} = -\{g, f\}$;
(ii) $\{f, g\}$ is linear in each of f and g;
(iii) $\{f, gk\} = \{f, g\}k + g\{f, k\}$;
(iv) $\{f, \{g, k\}\} + \{g, \{k, f\}\} + \{k, \{f, g\}\} = 0$.

Proof. The first two parts are immediate consequences of the definition, part (iii) follows from the product rule for differentiation, and (iv) can be derived by direct calculation. ♢

Corollary 1.3.2. Every function f satisfies the equation

$$\{f, f\} = 0.$$

Proof. By (i) $\{f, f\} = -\{f, f\}$, so $\{f, f\} = 0$. ♢

Corollary 1.3.3. If the Hamiltonian h does not depend explicitly on time then it is constant.

Proof. We have seen that

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \{h, h\}.$$

The first term on the right hand side vanishes because h does not depend explicitly on t, whilst the second vanishes by Corollary A1.4.2. ♢

This Corollary provides a generalization of the law of conservation of energy.

For later comparison with quantum theory it is useful to note that

$$\{x_j, x_k\} = 0, \qquad \{p_j, p_k\} = 0,$$

$$\{p_j, x_k\} = \delta_{jk}.$$