Calculus of One Variable

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0.0 Syllabus

1. Standard integrals. Integration by parts.
2. Motivation for ODEs. Order of an ODE. Separation of variables.
3. General linear homogeneous ODEs. Integrating factor for first order linear ODEs. Second solution when one solution known for second order linear ODEs.
4. First and second order linear ODEs with constant coefficients.
5. General solution of linear inhomogeneous ODE as particular solution plus solution of homogeneous equation. Simple examples of finding particular solutions by guesswork.
6. Elements of Matrix Theory: the calculation of determinants, eigenvalues and eigenvectors. Systems of linear coupled first order ODEs.

0.1 Recommended Texts

- D. W. Jordan & P. Smith, Mathematical Techniques, 3rd Edition, Oxford (2002) Chapters 17-20, 22.
- Erwin Kreyszig, Advanced Engineering Mathematics, 8th Edition, Wiley (1999).

0.2 Websites

http://www.maths.ox.ac.uk/∼ earl/
http://www.maths.ox.ac.uk/∼gaffney/

1. STANDARD INTEGRATION TECHNIQUES

1.1 Trigonometric Substitutions

We begin by looking at how to approach integrals such as

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{4-2x-x^2}} \quad \text{and} \quad \int_0^1 \frac{x\,\mathrm{d}x}{x^2+2x+2}.$$

You may recall similar but simpler integrals from A-level such as

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} \quad \text{and} \quad \int \frac{\mathrm{d}x}{1+x^2}.$$

To determine these we used trigonometric substitutions based on the identities

$$\sin^2\theta+\cos^2\theta=1, \qquad \tan^2\theta+1=\sec^2\theta, \qquad 1+\cot^2\theta=\csc^2\theta.$$

- **The principle being in each case that we would substitute for x in $1-x^2$ or in $1+x^2$ the correct trigonometric function to leave us with a single square**.

So in the first integral we could use $x=\sin\theta$ as $1-\sin^2\theta=\cos^2\theta$ and obtain

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} \quad =$$

Note that $x = \cos\theta$ would have worked just as well.

Some similar standard trigonometric integrals are

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} &= \sin^{-1} x + \text{const.} \\
\int \frac{\mathrm{d}x}{1+x^2} &= \tan^{-1} x + \text{const.} \\
\int \frac{\mathrm{d}x}{\sqrt{x^2+1}} &= \ln\left(x + \sqrt{x^2+1}\right) + \text{const.} \quad = \sinh^{-1} x + \text{const.} \\
\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} &= \ln\left|x + \sqrt{x^2-1}\right| + \text{const.} \quad = \cosh^{-1} x + \text{const.} \\
\int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} &= -\sqrt{1-x^2} + \text{const.} \\
\int \frac{x\,\mathrm{d}x}{\sqrt{x^2 \pm 1}} &= \sqrt{x^2 \pm 1} + \text{const.} \\
\int \frac{x\,\mathrm{d}x}{1+x^2} &= \frac{1}{2}\ln\left(1+x^2\right) + \text{const.}
\end{aligned}$$

If the fifth, sixth and seventh integrals are not apparent by inspection then a substitution

$$u = x^2 \quad \text{and noting } x = \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x}$$

may help.

For those with knowledge of the hyperbolic functions the third and fourth can be done more easily using the identity

$$\cosh^2 t = 1 + \sinh^2 t.$$

There is a printout available on the course website containing further details of the hyperbolic functions.

By completing the square (and making a substitution if desired) every integral of the form

$$\int \frac{Ax+B}{Cx^2+Dx+E}\,\mathrm{d}x \quad \text{and} \quad \int \frac{Ax+B}{\sqrt{Cx^2+Dx+E}}\,\mathrm{d}x$$

can be broken down into one or more of the previous forms. We return now to the integrals given at the start of the lecture.

Example 1 *Determine*

$$I_1 = \int_0^1 \frac{dx}{\sqrt{4-2x-x^2}} \quad \textit{and} \quad I_2 = \int_0^1 \frac{x\,dx}{x^2+2x+2}.$$

Solution.

$$I_1 =$$

The second integral I_2 appears on the first exercise sheet. ■

1.2 Integration by Parts

Integration by parts (IBP) can be used to tackle products of functions, but not just any product. Suppose we have an integral

$$\int f(x)\,g(x)\,\mathrm{d}x$$

in mind.

- **This will be approachable with IBP if one of these functions integrates/differentiates, perhaps repeatedly, to something simpler, whilst the other function differentiates/integrates to something of the same kind.**

Typically then $f(x)$ might be a polynomial which, after differentiating enough times, will become a constant; $g(x)$ on the other hand could be something like e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, all of which are functions which continually integrate to something similar. This remark reflects the nature of the formula for IBP which is:

Theorem 2 *(**Integration by Parts**) Let F and G be functions with derivatives f and g. Then*

$$\int F(x)\,g(x)\ \ dx = F(x)\,G(x) - \int f(x)\,G(x)\ \ dx.$$

IBP takes the integral of a product and leaves us with another integral of a product — but as we commented above, the point is that $f(x)$ should be a simpler function than $F(x)$ was whilst $G(x)$ should be no worse a function than $g(x)$ was.

Proof. The proof is simple — we just integrate the product rule of differentiation below, and rearrange

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)\,G(x)\right) = F(x)\,g(x) + f(x)\,G(x)\,.$$

■

Example 3 *Determine*

$$\int x^2 \sin x\ \ dx \quad \textit{and} \quad \int_0^1 x^3 e^{2x}\ \ dx.$$

Solution. Clearly x^2 will be the function that we need to differentiate down, and $\sin x$ is the function that will integrate in house. So we have, with *two* applications of IBP:

$$\int x^2 \sin x\ \mathrm{d}x \quad =$$

In a similar fashion

$$\begin{aligned}
\int_0^1 x^3 e^{2x}\ \mathrm{d}x &= \left[x^3\frac{e^{2x}}{2}\right]_0^1 - \int_0^1 3x^2\frac{e^{2x}}{2}\ \mathrm{d}x \quad \text{[IBP]} \\
&= \frac{e^2}{2} - \left(\left[3x^2\frac{e^{2x}}{4}\right]_0^1 - \int_0^1 6x\frac{e^{2x}}{4}\ \mathrm{d}x\right) \quad \text{[IBP]} \\
&= \frac{e^2}{2} - \frac{3e^2}{4} + \left[6x\frac{e^{2x}}{8}\right]_0^1 - \int_0^1 6\frac{e^{2x}}{8}\ \mathrm{d}x \quad \text{[IBP]} \\
&= \frac{-e^2}{4} + \frac{3e^2}{4} - \left[\frac{6e^{2x}}{16}\right]_0^1 \\
&= \frac{e^2}{8} + \frac{3}{8}.
\end{aligned}$$

■

This is by far the main use of IBP, the idea of eventually differentiating out one of the two functions. There are other important uses of IBP which don't quite fit into this type. These next two examples fall into the original class, but are a little unusual: in these cases we choose to integrate the polynomial factor instead as it is easier to differentiate the other factor. This is the case when we have a logarithm or an inverse trigonometric function as the second factor.

Example 4 *Evaluate*

$$\int (2x-1)\ln\left(x^2+1\right)\ dx \quad \textit{and} \quad \int \left(3x^2-4\right)\tan^{-1} x\ dx.$$

Solution. In both cases integrating the second factor looks rather daunting, certainly to integrate, but each factor differentiates nicely; recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x=\frac{1}{x}\quad\text{and that}\quad\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x=\frac{1}{1+x^2}.$$

So if we apply IBP to the above examples then we get

$$\int(2x-1)\ln\left(x^2+1\right)\ \mathrm{d}x=\left(x^2-x\right)\ln\left(x^2+1\right)-\int\left(x^2-x\right)\frac{2x}{x^2+1}\ \mathrm{d}x,$$

and

$$\int\left(3x^2-4\right)\tan^{-1}x\ \mathrm{d}x=\left(x^3-4x\right)\tan^{-1}x-\int\left(x^3-4x\right)\frac{1}{x^2+1}\ \mathrm{d}x.$$

To calculate the integrals

$$\int\frac{2x^3-2x^2}{x^2+1}\ \mathrm{d}x\quad\text{and}\quad\int\frac{x^3-4x}{x^2+1}\ \mathrm{d}x$$

we need to divide the denominator into the numerator and then use the previous list of standard integrals. For example, with the first, we note

$$2x^3-2x^2=(2x-2)\left(x^2+1\right)+(-2x+2)$$

and

$$\begin{aligned}\int\frac{2x^3-2x^2}{x^2+1}\ \mathrm{d}x &= \int\left(2x-2+\frac{-2x}{x^2+1}+\frac{2}{x^2+1}\right)\ \mathrm{d}x\\ &= x^2-2x-\ln\left(x^2+1\right)+2\tan^{-1}x+\text{const.}\end{aligned}$$

■

In the same vein as this we can use IBP to integrate functions which, at first glance, don't seem to be a product — this is done by treating a function $F(x)$ as the product $F(x)\times 1$.

Example 5 *Evaluate*

$$\int\ln x\ dx\ \text{and}\ \int\tan^{-1}x\ dx.$$

Solution. With IBP we see (integrating the 1 and differentiating the $\ln x$)

$$\int \ln x \ \mathrm{d}x \quad =$$

and similarly

$$\begin{aligned}\int \tan^{-1} x \ \mathrm{d}x &= \int 1 \times \tan^{-1} x \ \mathrm{d}x \\ &= x\tan^{-1} x - \int x \frac{1}{1+x^2} \ \mathrm{d}x \\ &= x\tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right) + \text{const.}\end{aligned}$$

■

Sometimes both functions remain *in house,* but we eventually return to our original integrand.

Example 6 *Determine*

$$\int e^x \sin x \ dx.$$

Solution. Both of these functions now remain in house, but if we apply IBP twice, integrating the e^x and differentiating the $\sin x$, then we see

$$\int e^x \sin x \ \mathrm{d}x \quad = \quad e^x \sin x - \int e^x \cos x \ \mathrm{d}x \quad =$$

We see that we have returned to our original integral, and so we can rearrange this equality to get

$$\int e^x \sin x \, \mathrm{d}x = \frac{1}{2} e^x (\sin x - \cos x) + \text{const.}$$

■

When IBP needs to be applied repetitively to determine an integral it can often make sense to consider the general case and set up a **reduction formula**. For example, in order to calculate

$$\int \cos^7 \theta \, d\theta$$

we first set

$$I_n = \int \cos^n \theta \, \mathrm{d}\theta,$$

and we will aim to write I_n in terms of other I_k where $k < n$, eventually reducing the problem to calculating I_0, or I_1 say, which are simple integrals.

Using IBP we see

$$I_n = \int \cos^{n-1} \theta \times \cos \theta \, \mathrm{d}\theta =$$

Rearranging this we see

$$I_n = \frac{\cos^{n-1}\theta\sin\theta}{n} + \frac{n-1}{n} I_{n-2}.$$

With this reduction formula I_n can be rewritten in terms of simpler and simpler integrals until we are left only needing to calculate I_0, if n is even, or I_1, if n is odd — both these integrals are easy to calculate.

Example 7 *Calculate*

$$I_7 = \int \cos^7\theta \; d\theta.$$

Solution. Using the reduction formula above

$$\begin{aligned}
I_7 &= \frac{\cos^6\theta\sin\theta}{7} + \frac{6}{7}I_5 \\
&= \frac{\cos^6\theta\sin\theta}{7} + \frac{6}{7}\left(\frac{\cos^4\theta\sin\theta}{5} + \frac{4}{5}I_3\right) \\
&= \frac{\cos^6\theta\sin\theta}{7} + \frac{6\cos^4\theta\sin\theta}{35} + \frac{24}{35}\left(\frac{\cos^2\theta\sin\theta}{3} + \frac{2}{3}I_1\right) \\
&= \frac{\cos^6\theta\sin\theta}{7} + \frac{6\cos^4\theta\sin\theta}{35} + \frac{24\cos^2\theta\sin\theta}{105} + \frac{48}{105}\sin\theta + \text{const.}
\end{aligned}$$

■

Alternative Method (for those with some knowledge of complex numbers). Recall $2\cos\theta = e^{i\theta} + e^{-i\theta}$ so that

$$\begin{aligned}
128\cos^7\theta &= e^{7i\theta} + 7e^{5i\theta} + 21e^{3i\theta} + 35e^{i\theta} + 35e^{-i\theta} + 21e^{-3i\theta} + 7e^{-5i\theta} + e^{-7i\theta} \\
&= 2\cos 7\theta + 14\cos 5\theta + 42\cos 3\theta + 70\cos\theta.
\end{aligned}$$

Then

$$\begin{aligned}\int \cos^7 \theta \ \mathrm{d}\theta &= \int \left(\frac{1}{64} \cos 7\theta + \frac{7}{64} \cos 5\theta + \frac{21}{64} \cos 3\theta + \frac{35}{64} \cos \theta \right) \ \mathrm{d}\theta \\ &= \frac{1}{448} \sin 7\theta + \frac{7}{320} \sin 5\theta + \frac{7}{64} \sin 3\theta + \frac{35}{64} \sin \theta + \text{const.}\end{aligned}$$

Example 8 *Calculate*

$$\int_0^1 x^3 e^{2x} \ dx$$

Solution. We already met this integral in Example 3. We can approach this in a simpler, yet more general, fashion by setting up a reduction formula. For a natural number n, let

$$J_n = \int_0^1 x^n e^{2x} \ \mathrm{d}x.$$

We can use integration by parts to show

$$\begin{aligned}J_n &= \left[x^n \frac{e^{2x}}{2} \right]_0^1 - \int_0^1 n x^{n-1} \frac{e^{2x}}{2} \ \mathrm{d}x \\ &= \frac{e^2}{2} - \frac{n}{2} J_{n-1} \quad \text{if } n \geqslant 1\end{aligned}$$

which is our reduction formula. We first note

$$J_0 = \int_0^1 e^{2x} \ \mathrm{d}x = \left[\frac{e^{2x}}{2} \right]_0^1 = \frac{e^2 - 1}{2},$$

and then applying the reduction formula the calculations made in Example 3 looks so much easier on the eye:

$$\begin{aligned}J_3 &= \frac{e^2}{2} - \frac{3}{2} J_2 \\ &= \frac{e^2}{2} - \frac{3}{2} \left(\frac{e^2}{2} - \frac{2}{2} J_1 \right) \\ &= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2} \left(\frac{e^2}{2} - \frac{1}{2} J_0 \right) \\ &= \frac{e^2}{8} + \frac{3}{8}.\end{aligned}$$

■

Some integrands may involve two variables, such as:

Example 9 *Calculate for positive integers m, n the integral*

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \ dx.$$

Solution. Calculating either $B(m,1)$ or $B(1,n)$ is easy; for example

$$B(m,1) = \int_0^1 x^{m-1} \, \mathrm{d}x = \frac{1}{m}. \tag{1.1}$$

So it would seem best to find a reduction formula that moves us towards either of the integrals $B(m,1)$ or $B(1,n)$. Using integration by parts, if $n \geqslant 2$ we have

$$\begin{aligned} B(m,n) &= \left[\frac{x^m}{m}(1-x)^{n-1}\right]_0^1 - \int_0^1 \frac{x^m}{m} \times (n-1) \times (-1)(1-x)^{n-2} \ \mathrm{d}x \\ &= \end{aligned}$$

So if $n \geqslant 2$ we can apply this to see

$$\begin{aligned} B(m,n) &= \frac{n-1}{m} B(m+1,n-1) \\ &= \frac{n-1}{m} \times \frac{n-2}{m+1} B(m+2,n-2) \\ &= \end{aligned}$$

Equation (1.1) shows this formula also holds for $n = 1$. ■

2. DIFFERENTIAL EQUATIONS

2.1 Introduction and History

The study of ordinary differential equations (DEs) is as old as calculus itself and dates back to the time of Newton (1643-1727) and Leibniz (1646-1716). At that time most of the interest in DEs came from applications in physics and astronomy — one of Newton's greatest achievements, in his *Principia Mathematica* (1687), was to show that a force between a planet and the sun, which is inversely proportional to the square of the distance between them, would lead to an elliptical orbit.

The study of differential equations grew as increasingly varied mathematical and physical situations led to differential equations, and as more and more sophisticated techniques were found to solve them. Besides in astronomy DEs began appearing naturally in applied areas such as fluid dynamics, heat flow, waves on strings, in determining the curve a chain between two points will make under its own weight, and equally in pure mathematics, finding the shortest path between two points on a surface, the surface across a given boundary of smallest area (i.e. the shape of a soap film), the largest area a curve of fixed length can bound, etc.

Still today much of applied mathematics is concerned with the solution of differential equations that arise from the modelling of real world situations. There is much current interest in financial mathematics where the variables present in the differential equations — equations aiming to model the stock exchange for example — are no longer certain but are modelled with random variables. Such equations are known as *stochastic* differential equations.

In this first course we will be studying **ordinary differential equations** (ODEs) rather than **partial differential equations** (PDEs). This means that the DEs in question will involve *full* derivatives, such as $\mathrm{d}y/\mathrm{d}x$, rather than *partial* derivatives, such as $\partial y/\partial x$. The latter notation is a measure of how a function y changes with x whilst all other variables (which y depends on) are kept constant. We will meet partial derivatives in the second course.

We give here, and solve, a simple example which involves some of the key ideas of DEs; the example here is the movement of a particle P under gravity, in one vertical dimension. Our starting point is Newton's second law on neglect of air resistance:

$$\text{Force} = \text{mass} \times \text{acceleration} = \text{mass} \times (-g),$$

where $g \sim 9.8$ m s^{-2} is the magnitude of the gravitational acceleration. Note the need for a minus sign here as gravity is acting downwards. Writing $h(t)$ for the height (in meters, say) of P over the ground, the **velocity** of the particle is the quantity $\mathrm{d}h/\mathrm{d}t$ — the rate of change of distance (here, height) with time. The rate of change of velocity with time is called **acceleration** and is the quantity $\mathrm{d}^2h/\mathrm{d}t^2$. Thus

(2.1)

Equation (2.1) is not a difficult DE to solve; we can integrate first once,

(2.2)

and then again

(2.3)

where K_1 and K_2 are constants. Currently we don't know enough about the specific case of the particle P to be able to say anything more about these constants. Note though that whatever the values of K_1 and K_2 the graph of h against t is a parabola.

Definition 10 *An* ***ordinary differential equation*** *is a equation relating a function, say y, in one variable, say x, and finitely many of its derivatives. i.e. something that can be written in the form*

$$f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ky}{dx^k}\right) = 0$$

for some function f and some natural number k. Here x is the ***independent variable*** *and the DE governs how the* ***dependent variable*** *y varies with x. The equation may have no or many functions $y(x)$ which satisfy it; the problem usually is to find the most general form of* ***solution*** *$y(x)$ of function which satisfies the differential equation.*

Definition 11 *A derivative of the form d^ky/dx^k is said to be of order k and we say that a DE has* ***order*** *k if it involves derivatives of order k and less.*

Example 12 *The following are types and famous examples of ordinary differential equations.*

- A **first order differential equation** is one of the form
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

- A k**th order inhomogeneous linear differential equation** is one of the form

$$a_k(x)\frac{\mathrm{d}^k y}{\mathrm{d}x^k}+a_{k-1}(x)\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}}+\cdots+a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x}+a_0(x)\,y=f(x),$$

  and the equation is called **homogeneous** if $f(x)=0$.

- The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y, \qquad y(0)=1$$

  uniquely characterises the function $y=e^x$.

- The equation for simple harmonic motion is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}=-\omega^2 y$$

  and the DE governing the swinging of a pendulum is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=-\frac{g}{l}\sin\theta.$$

- **Bessel's equation**, which relates to vibrations in a circular membrane is

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(x^2-\nu^2\right)y=0.$$

- **Legendre's equation,** which relates to **Laplace's equation** which we will meet in the second course, is

$$\left(1-x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+m(m+1)\,y=0.$$

We return now to equation (2.1), which is a second order DE. In some loose sense solving a DE of order k involves integrating k times, though not usually in such an obvious fashion as in the case of (2.1). So we would expect the solution of an order k DE to have k undetermined constants in it, and this will be the case in most of the simple examples that we look at here. However this is not generally the case and we will see other examples where more, or fewer, than k constants are present in the solution.

The expression (2.3) is the **general solution** for the solution of the DE (2.1), that is an expression which encompasses by means of indeterminates, K_1 and K_2, all solutions of the DE.

At the moment then

$$h(t) = -\frac{1}{2}gt^2 + K_1 t + K_2$$

is not unique, but rather depends on two undetermined constants. And this isn't unreasonable as the particle P could follow many a path; at the moment we don't have enough information to characterise the path uniquely.

One way of filling in the missing info would be to say how high P was at $t = 0$ and how fast it was going at that point. For example, suppose P started at a height of 100m and we threw it up into the air at a speed of 10ms^{-1} — that is

$$h(0) = 100 \quad \text{and} \quad \frac{\mathrm{d}h}{\mathrm{d}t}(0) = 10. \tag{2.4}$$

Putting these values into equations (2.2) and (2.3) we'd get

Thus the height of P at time t has been uniquely determined and is given by

$$h(t) = 100 + 10t - \frac{1}{2}gt^2.$$

The extra bits of information given in equation (2.4) are called **initial conditions** — the DE (2.1) with the initial conditions (2.4) is called an **initial-value problem.**

Alternatively, suppose we were told that P was thrown at time $t = 0$ from a height of 100m and was subsequently one second later caught at 105m. That is

$$h(0) = 100 \quad \text{and} \quad h(1) = 105. \tag{2.5}$$

Putting these values into the general solution (2.3) gives us

$$\begin{aligned} K_2 &= 100, \\ \frac{-1}{2}g + K_1 + K_2 &= 105 \Longrightarrow K_1 = 5 + \frac{g}{2}. \end{aligned}$$

Hence

$$h(t) = \frac{-1}{2}gt^2 + \left(5 + \frac{g}{2}\right)t + 100.$$

Again we have uniquely characterised the trajectory of P by saying where the particle is at two times. The conditions (2.5) are called **boundary conditions** and the DE (2.1) with the boundary conditions (2.5) is called a **boundary-value problem.**

Having solved the earlier initial-value problem and found an equation for h then we could easily answer other questions about P's behaviour such as

- what is the greatest height P achieves? The maximum height will be a stationary value for $h(t)$ and so we need to solve the equation $h'(0) = 0$, which has solution $t = 10/g$. At this time the height is

$$h(10/g) = 100 + \frac{100}{g} - \frac{100g}{2g^2} = 100 + \frac{50}{g}.$$

- what time does P hit the ground?

  One of these times is meaningless (being negative, and so before our experiment began) and so we take the other (positive) solution and see that P hits the ground at

$$t = \frac{10 + 10\sqrt{1+2g}}{g}.$$

The next example is designed to show that we should not be cavalier when solving DEs.

Example 13 *Find the general solution of the DE*

$$\left(\frac{dy}{dx}\right)^2 = 4y. \tag{2.6}$$

Solution. Given this equation we might argue as follows — taking square roots we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{y} \quad \Longrightarrow \quad \frac{1}{2\sqrt{y}}\frac{\mathrm{d}y}{\mathrm{d}x} = 1. \tag{2.7}$$

From the chain rule we recognise the LHS as the derivative of $\sqrt{y}$, and so, integrating with respect to (wrt) x we have $\sqrt{y} = x + K$, where K is a constant. Squaring this, we might think that the general solution has the form $y = (x + K)^2$.

What, if anything, could have gone wrong with this argument?

We could have been more careful to include positive and negative square roots at the (2.7) stage, but actually we don't lose any solutions by this oversight. Thinking a little more, we might realise that we have missed the most obvious of solutions: the zero function, $y = 0$, which isn't present in our 'general' solution. At this point we might scold ourselves for committing the crime of dividing by zero at stage (2.7), rather than treating $y = 0$ as a separate case. But we have lost many more than just one solution at this point here by being careless. The general solution of (2.6) is in fact

$$y(x) =$$

where a and b are constants satisfying $-\infty \leqslant a \leqslant b \leqslant \infty$. We missed whole families of solutions by being careless — note also that the general solution requires *two* constants in its description even though the DE is only first order.

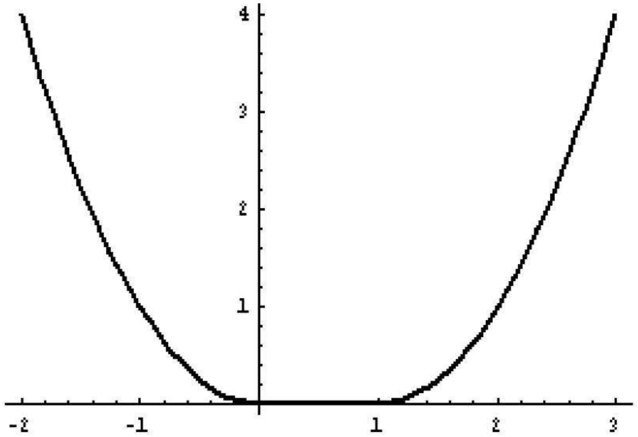


Figure 2-1 A missed solution with $a = 0$, $b = 1$.

■

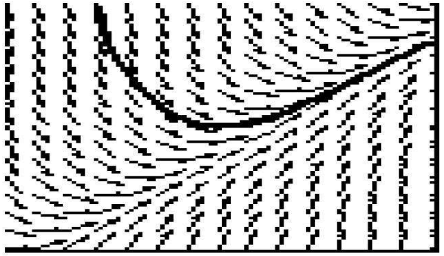
2.2 Graphical Considerations

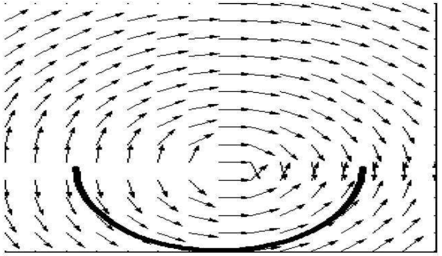
Consider the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x - y.$$

By the end of Section 3.1 we will be able to solve such equations using *Integrating Factors*. The general solution is of the form $y = x - 1 + Ae^{-x}$. But even without this knowledge we can perform some qualitative analysis on what the solutions might look like.

In the first diagram below is a plot of the direction field of the DE — that is at each point (x, y) in the diagram is drawn a short vector (here of unit length) whose gradient equals $x - y$.





The direction field is plotted for $-3 < x, y < 3$ and the solution $y = x - 1 + e^{-x}$ is represented in bold. Even though we are not currently able to solve the DE we can see that solutions typically remain on one side of the line $y = x - 1$ and tend towards this line as x increases. The second direction field is for the DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y+1}$$

which we will solve in the next section. The solution that is plotted is $y = -1 - \sqrt{4 - x^2}$, which is valid for $-2 < x < 2$. This is the solution to DE with initial condition $y(0) = -3$, as we will see. Again, even though we have not yet solved the DE, the solutions' semi-circular nature is clear and also that the solutions stay on either side of the line $y = -1$. As each solution approaches the line $y = -1$ we see that there is nowhere further that we can extend the solution.

2.3 Separable Equations

Recall that a first order differential equation has the general form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

A special class of first order differential equation contains those of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)\, b(y)$$

and these are known as **separable equations**. Such equations can, in principle, be easily rearranged and solved as follows:

$$\frac{1}{b\left(y\right)}\frac{\mathrm{d}y}{\mathrm{d}x}=a\left(x\right) \tag{2.8}$$

and then integrating with respect to x, we find

$$\int\frac{\mathrm{d}y}{b\left(y\right)}=\int a\left(x\right)\ \mathrm{d}x. \tag{2.9}$$

Remark 14 *In some texts, or previously at A-level, equation (2.8) may be written*

$$\frac{dy}{b\left(y\right)}=a\left(x\right)\ \ dx \tag{2.10}$$

and subsequently integrated. At first glance, though, this makes no rigorous sense. dy/dx *exists as the limit of an approximating gradient* $\delta y/\delta x$ *whereas in the limit* δx *and* δy *are both* 0*. So (2.10) seems to say nothing more than* $0=0$*. However the equation can be made rigorous with knowledge of differentials (which are beyond the scope of this course) and no error will result from an equation like (2.10), but we should recognise, with our current definition of* dy/dx*, that the equation has no rigorous meaning.*

Remark 15 *Equation (2.9) may offer some practical problems as a solution to the differential equation. Firstly the functions* $1/b$ *and* a *may not be readily integrable, if at all. But further, even if they both are nicely integrable, to end with a solution of the form*

$$B\left(y\right)=A\left(x\right)+const.$$

is this really solving the differential equation as asked? We originally said a solution was a function $y\left(x\right)$ *which satisfies the differential equation, and our solution is not in this form, and may be difficult to put into this form.*

Example 16 *(Exponential Growth and Decay) In many examples from science the rate of change of a variable is proportional to the variable itself:*

- *growth in a bacterial sample;*
- *a capacitor discharging;*
- *radioactive decay.*

That the rate of change with time t of a quantity, say y, is proportional to y is encoded in the differential equation

$$\frac{dy}{dt} = ky$$

where k is a constant. If we separate the variables we find

$$\frac{1}{y}\frac{dy}{dt} = k$$

and integrating with respect to t we have, for some constant C,

$$\ln y = kx + C \Longrightarrow y = Ae^{kx} \tag{2.11}$$

where A is another constant. This is the general solution of the given differential equation, where A is any real number. Note that we can characterise our solution uniquely with an initial condition. In fact the exponential function is characterised as the one solution of the initial value problem

$$\frac{dy}{dx} = y, \qquad y(0) = 1.$$

Remark 17 *This presentation of the solution is rather "A-level". Two points have been overlooked. Firstly what if $y = 0$? Well, we can deal with the $y = 0$ case by treating it as a separate case. Secondly though is to note that $y = Ae^{kx}$ is a solution when $A < 0$ yet here $A = e^C$ seems always to be positive. And for the remaining argument we should really write*

$$\frac{1}{y}\frac{dy}{dt} = k \Longrightarrow$$

as $\ln|y|$ is an indefinite integral of the LHS whether or not y is positive. Then

$$|y| =$$

gives the correct general solution.

Example 18 *Find the general solution to the separable differential equation*

$$\sin x \frac{dy}{dx} = y \ln y.$$

Solution. Separating variables we find

Integrating with respect to x we get

which rearranges to

where A is another constant. The solution is valid in the range $-\pi < x < \pi$. ■

Exercise 1 *Which of these solutions satisfy the initial condition* $y(0) = 1$*? How many solutions satisfy the initial condition* $y(0) = 2$*? Why are these answers unsurprising when we look at the original differential equation?*

Example 19 *Find the solution of the initial value problem*

$$\frac{dy}{dx} = \frac{-x}{y+1}, \qquad y(0) = -3. \tag{2.12}$$

Solution. The equation is separable and so we may rearrange it to

$$(y+1)\frac{\mathrm{d}y}{\mathrm{d}x} = -x$$

and integrate to find

$$\frac{1}{2}(y+1)^2 = -\frac{1}{2}x^2 + C.$$

As $y = -3$ when $x = 0$ then $C = 2$ and we have

$$x^2 + (y+1)^2 = 4, \tag{2.13}$$

which is the equation of a circle.

However by definition a solution of a differential equation is a function y of x whereas for each x in the range $-2 < x < 2$ there are two y which satisfy (2.13). But any solution $y(x)$ of (2.12) must satisfy (2.13). So we can solve for y obtaining

$$y = -1 - \sqrt{4 - x^2}$$

noting we need to take the negative root as $y(0) = -3$. Note also that the solution is valid in the range $-2 < x < 2$. ■

Example 20 *Find the solution of the initial value problem*

$$\left(5y^4 - 1\right)\frac{dy}{dx} = 1, \qquad y\left(0\right) = 1.$$

Solution. We can straight away integrate both sides wrt x and obtain

Putting $x = 0$ and $y = 1$ we see that $C = 0$ and so

$$\tag{2.14}$$

This equation does not give us a simple expression for the solution y in terms of x, but the solution y has to satisfy equation (2.14). Sketch the curve $y^5 - y = x$:

We see that the solution y, which goes through the point $\left(0, 1\right)$, can be extended arbitrarily to greater x but can only be extended down to $\left(\frac{-4}{5\sqrt[4]{5}}, \frac{1}{\sqrt[4]{5}}\right)$ where $\mathrm{d}y/\mathrm{d}x$ becomes infinite — there are no values we can assign to y for $x < \frac{-4}{5\sqrt[4]{5}}$ without having a discontinuity in the solution y.

On the other hand, we cannot solve equation (2.14) exactly for given x and it remains our only way to describe the solution. It should be appreciated though that the whole curve is not the solution but just the part which appears in bold on the graph.

2.4 Equations Adjustable to Separable Equations

By a **homogeneous polar** differential equation we will mean one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right). \tag{2.15}$$

- **Note these DEs are often simply called homogeneous, but we will use the term homogeneous polar here to distinguish them from other DEs we will meet later and refer to as homogeneous.**

These can be solved with a substitution of the form

$$y(x) = v(x)\,x \tag{2.16}$$

to get a new equation in terms of x and the dependent variable v. Note from the product rule of differentiation that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and so making the substitution (2.16) into the DE (2.15) gives us the new DE

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = f(v) - v,$$

which is a separable DE.

Example 21 *Find the general solution of the DE*

$$\frac{dy}{dx} = \frac{x+4y}{2x+3y}.$$

Solution. At first glance this may not look like a homogeneous polar DE, but we can see this to be the case by rewriting the RHS as

$$\frac{1+4\frac{y}{x}}{2+3\frac{y}{x}}.$$

If we make the substitution $y(x) = xv(x)$ then we have

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+4v}{2+3v}.$$

Rearranging this gives

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+4v}{2+3v} - v = \frac{1+2v-3v^2}{2+3v}$$

and so, separating the variables, we find

$$\int \frac{2+3v}{1+2v-3v^2}\ \mathrm{d}v = \int \frac{\mathrm{d}x}{x}.$$

Using partial fractions gives

$$= \frac{1}{4}\int \left(\frac{5}{1-v} + \frac{3}{1+3v}\right)\ \mathrm{d}v = \ln|x| + C,$$

and resubstituting $v = y/x$ leads us to the general solution

$$-\frac{5}{4}\ln|1-v| + \frac{1}{4}\ln|1+3v| = \ln|x| + C.$$

This can be simplified somewhat to

$$x + 3y = A\,(x-y)^5$$

where A is a constant. ■

Example 22 *Solve the initial value problem*

$$\frac{dy}{dx} = \frac{y}{x+y+2}, \qquad y\,(0) = 1. \tag{2.17}$$

Solution. This DE is not homogeneous polar, but it can easily be made into such a DE with a suitable change of variables. We introduce new variables

$$X = x + a \quad \text{and} \quad Y = y + b,$$

with the aim to choose a and b so that the DE becomes homogeneous polar. If we make these substitutions then the RHS equals

which is homogeneous polar if $b = 0$ and $2 - a - b = 0$ i.e. $a = 2$. With these values of a and b, noting that $\mathrm{d}Y/\mathrm{d}X = \mathrm{d}y/\mathrm{d}x$ and that the initial condition has become $Y\,(X=2) = Y\,(x=0) = y\,(x=0) = 1$, our initial value problem now reads as

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \qquad\qquad\qquad\qquad Y\,(2) = \qquad\qquad .$$

Substituting in $Y = VX$ gives us

$$V + X\frac{\mathrm{d}V}{\mathrm{d}X} = \qquad\qquad\qquad\qquad V(2) = \qquad\qquad .$$

Rearranging and separating the variables gives

Substituting in our initial condition, $V(2) = 1/2$, we see that $K = 2$ and so

Noting $V = Y/X$, and with some rearranging we have

$$X - Y\ln Y = 2Y \qquad X, Y > 0.$$

Further, as $X = x + 2$ and $Y = y$, our solution to the initial value problem (2.17) has become

$$x + 2 = 2y + y\ln y \qquad x > -2, y > 0.$$

■

Example 23 *By means of a substitution transform the following into a separable equation and find its general solution:*

$$\frac{dy}{dx} = \cos(x + y).$$

Solution. This is neither separable, nor homogeneous polar, but the substitution

$$z = x + y$$

would seem a sensible one to simplify the RHS. We might then hope to get a separable equation in x and z. As $y = z - x$ then

$$\frac{\mathrm{d}z}{\mathrm{d}x} - 1 = \cos z.$$

This is separable and we find

$$\int \frac{\mathrm{d}z}{\cos z + 1} = x + C.$$

Integrating the LHS is made easier with the trigonometric identity $\cos z + 1 = 2\cos^2(z/2)$, giving

$$\tan\frac{z}{2} = \frac{1}{2}\int \sec^2\left(\frac{z}{2}\right)\,\mathrm{d}z = x + C.$$

So

$$\tan\frac{x+y}{2} = x + C$$

and rearranging gives

$$y = 2\tan^{-1}(x + C) - x, \qquad x \in \mathbb{R}$$

as the general solution. ■

3. LINEAR DIFFERENTIAL EQUATIONS

An important class of differential equation consists of the linear differential equations. These are important because of their theory, because a great many important ODEs are linear, and also because a linear differential equation can sometimes be used to approximate a non-linear equation.

Definition 24 *An* ***inhomogeneous linear DE*** *of order k is one of the form*

$$a_k(x)\frac{d^ky}{dx^k}+a_{k-1}(x)\frac{d^{k-1}y}{dx^{k-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)\,y=f(x)$$

where $a_k(x)\neq 0$. If $f(x)=0$ then the DE is called ***homogeneous linear****.*

3.1 Integrating Factors

A first order homogeneous linear differential equation is one of the form

$$P(x)\frac{\mathrm{d}y}{\mathrm{d}x}+Q(x)\,y=0$$

and we can see that this DE is also separable — we have already tackled such DEs. A first order inhomogeneous linear DE is of the form

$$P(x)\frac{\mathrm{d}y}{\mathrm{d}x}+Q(x)\,y=R(x)$$

and these can be approached by using **Integrating Factors**. The idea is to multiply the LHS by a factor which will make it the derivative of a product of the form $A(x)\,y$.
Consider the first order inhomogeneous linear DE

$$P(x)\frac{\mathrm{d}y}{\mathrm{d}x}+Q(x)\,y=R(x)\,. \tag{3.1}$$

In general, the LHS of (3.1) won't be expressible as the derivative of a product $A(x)\,y$. However, if we multiply both sides of the DE by an appropriate **Integrating Factor** $I(x)$ then we can turn the LHS into the derivative such a product.

Let's first of all simplify the equation by dividing through by $P(x)$, and then multiplying by an integrating factor $I(x)$ (which we have yet to determine) to get

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)\frac{Q(x)}{P(x)}y = I(x)\frac{R(x)}{P(x)}. \tag{3.2}$$

We would like the LHS to be the derivative of a product $A(x)\,y$ for some function $A(x)$ — from the product rule $A(x)\,y$ differentiates to

$$A(x)\frac{\mathrm{d}y}{\mathrm{d}x} + A'(x)\,y. \tag{3.3}$$

So equating the coefficients of y and y' in (3.2) and (3.3), we have

$$A(x) = I(x) \quad \text{and } A'(x) = \frac{I(x)\,Q(x)}{P(x)}.$$

Rearranging this gives

$$\frac{I'(x)}{I(x)} = \frac{Q(x)}{P(x)}.$$

The LHS is the derivative of $\ln I(x)$ and so we see

$$I(x) = \exp\int\frac{Q(x)}{P(x)}\,\mathrm{d}x.$$

We are only looking for one such $I(x)$ with this property; we do not need to worry about the constant of integration.

For this choice of $I(x)$, (3.2) now reads as

$$\frac{\mathrm{d}}{\mathrm{d}x}(I(x)\,y) = \frac{I(x)\,R(x)}{P(x)}$$

which has the general solution

$$y(x) = \frac{1}{I(x)}\left(\int\frac{I(x)\,R(x)}{P(x)}\,\mathrm{d}x + \text{const.}\right).$$

Example 25 *Find the general solution of the DE*

$$x\frac{dy}{dx} + (x-1)\,y = x^2.$$

Solution. If we divide through by x we get

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-\frac{1}{x}\right)y=x$$

and we see that the integrating factor is

$$I\left(x\right) \quad =$$

Multiplying through by the integrating factor gives

which, by construction, rearranges to

Integrating gives

$$\frac{1}{x}e^{x}y=e^{x}+K$$

where K is a constant, and rearranging gives

$$y\left(x\right)=x+Kxe^{-x}$$

as our general solution. ■

Example 26 *Solve the initial value problem*

$$\frac{dy}{dx}+2xy=1, \quad y(0)=0.$$

Solution. The integrating factor here is

$$I(x)=\exp\int 2x \ \mathrm{d}x=\exp\left(x^2\right).$$

Multiplying through we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}y\right)=e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x}+2xe^{x^2}y=e^{x^2}.$$

Noting that $y(0)=0$, when we integrate this we arrive at

$$e^{x^2}y=\int_0^x e^{t^2} \ \mathrm{d}t,$$

and rearranging gives

$$y(x)=e^{-x^2}\int_0^x e^{t^2} \ \mathrm{d}t.$$

The integral of e^{x^2} can't be expressed in a closed form involving elementary equations (hence the need for normal distribution tables etc.) and we have to leave the answer in the given form. ■

Example 27 *Solve the initial value problem*

$$y\frac{dy}{dx}+\sin x=y^2, \qquad y(0)=1.$$

Solution. This DE is neither linear nor separable. However if we note that

$$y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right)$$

then we see that the substitution $z=y^2$ turns the given DE into

$$\frac{\mathrm{d}z}{\mathrm{d}x}-2z=-2\sin x, \qquad z(0)=1^2=1$$

which is solvable by integrating factors. In this case the integrating factor is e^{-2x} and we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{-2x}\right)=-2e^{-2x}\sin x.$$

Integrating the RHS by parts as in Example 6 we get

$$ze^{-2x} = \frac{e^{-2x}}{5}\left(4\sin x + 2\cos x\right) + C.$$

As $z = 1$ when $x = 0$ then $C = 3/5$ and so, recalling that $z = y^2$, we have

$$y = \sqrt{\frac{4\sin x + 2\cos x + 3e^{2x}}{5}},$$

taking the positive root, and with the solution being valid on the interval containing 0 for which $4\sin x + 2\cos x + 3e^{2x} > 0$. ■

3.2 Second Order Homogeneous Linear Differential Equations

A second order homogeneous linear differential equation is of the form

$$P(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)\,y = 0. \tag{3.4}$$

We shall consider the situation where, either by inspection or other means, we already know of a solution $Y(x)$. Knowing this it is possible to transform the above equation into a first order differential equation.

We begin by making the substitution

$$y(x) = Y(x)\,z(x)$$

thus turning (3.4) into a DE involving z and x. Note

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \qquad\qquad\qquad\qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} =$$

Substituting these expressions into (3.4) we find

$$P(x)\left(Y(x)\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + 2\frac{\mathrm{d}Y}{\mathrm{d}x}\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}^2 Y}{\mathrm{d}x^2}z\right) + Q(x)\left(Y(x)\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}Y}{\mathrm{d}x}z\right) + R(x)\,Y(x)\,z(x) = 0$$

which rearranges to

$$\tag{3.5}$$

Now the bracket

$$P(x)\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}+Q(x)\frac{\mathrm{d}Y}{\mathrm{d}x}+R(x)\,Y(x)$$

equals zero as we know that Y is a solution to (3.4). Further if we let $w(x)=\mathrm{d}z/\mathrm{d}x$ then we can see (3.5) is a first order separable DE in w:

$$P(x)\,Y(x)\frac{\mathrm{d}w}{\mathrm{d}x}+\left(2P(x)\frac{\mathrm{d}Y}{\mathrm{d}x}+Q(x)\,Y(x)\right)w=0 \tag{3.6}$$

which is solvable to find w, which we may integrate to find z and so y

Example 28 *Show that $u(x)=1/x$ is a solution of*

$$x\frac{d^2y}{dx^2}+2(1-x)\frac{dy}{dx}-2y=0.$$

Hence find the equation's general solution.

Solution. It is easy to check that $Y(x)=1/x$ is a solution as

$$x\left(\frac{2}{x^3}\right)+2(1-x)\left(\frac{-1}{x^2}\right)-2\left(\frac{1}{x}\right)=0.$$

For this example equation (3.6) reads as where $w = \mathrm{d}z/\mathrm{d}x$ and $z\,(x)\,/x = y\,(x)$. Simplifying we have

$$\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}x} = 2,$$

which we know from Example 16 to have general solution

$$w = Ae^{2x}.$$

Integrating we have

$$z =$$

(where $a = A/2$) and hence

$$y = zY =$$

is the original DE's general solution. ■

Example 29 *Show that Legendre's equation (see Example 12) with* $m = 1$

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

has a solution of the form $Y\,(x) = ax + b$ *and hence determine its general solution.*

Solution. The function $Y\,(x) = ax + b$ is a solution of the DE if

$$\left(1 - x^2\right)(0) - 2x\,(a) + 2\,(ax + b) = 0.$$

Equating coefficients we see that

$$-2a + 2a = 0; \qquad 2b = 0.$$

These show $b = 0$ and that a can be anything; in particular $Y(x) = x$ is a solution. For this example, equation (3.6) reads as

$$\left(1 - x^2\right)(x)\frac{\mathrm{d}w}{\mathrm{d}x} + \left(\left(2 - 2x^2\right)(1) + (-2x)(x)\right)w = 0,$$

where $w = \mathrm{d}z/\mathrm{d}x$ and $z = y/x$. The above simplifies to

$$\left(x - x^3\right)\frac{\mathrm{d}w}{\mathrm{d}x} + \left(2 - 4x^2\right)w = 0.$$

Separating variables we find

$$\int \frac{\mathrm{d}w}{w} = \int \frac{2 - 4x^2}{x^3 - x}\,\mathrm{d}x = \int \left(\frac{-2}{x} + \frac{-1}{x-1} + \frac{-1}{x+1}\right)\mathrm{d}x,$$

giving

$$\ln|w| = -2\ln|x| - \ln|x-1| - \ln|x+1| + C,$$

and

$$w = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{A}{x^2(x-1)(x+1)} = A\left(\frac{-1}{x^2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right).$$

Hence the general solution is

$$y(x) = xz(x) = A + \frac{Ax}{2}\ln\left|\frac{x-1}{x+1}\right| + Bx.$$

■

3.3 The Linear Algebra behind Linear Differential Equations

Recall that a homogeneous linear DE of order k is one of the form

$$a_k(x)\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0.$$

As you will meet in other areas of mathematics, especially in the Linear Algebra courses, the space of solutions has some nice algebraic properties.

Theorem 30 *Let y_1 and y_2 be solutions of a homogeneous linear differential equation and α_1, α_2 be real numbers. Then $\alpha_1 y_1 + \alpha_2 y_2$ is also a solution of the DE. Note also that the zero function is always a solution. This means that the space of solutions of the DE is a* ***real vector space****.*

Proof. We know that

$$\begin{aligned} a_k(x)\frac{\mathrm{d}^k y_1}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1} y_1}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y_1}{\mathrm{d}x} + a_0(x)\,y_1 &= 0, \qquad (3.7)\\ a_k(x)\frac{\mathrm{d}^k y_2}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1} y_2}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y_2}{\mathrm{d}x} + a_0(x)\,y_2 &= 0. \qquad (3.8)\end{aligned}$$

If we add α_1 times equation (3.7) to α_2 times equation (3.8) and rearrange we find

$$a_k(x)\frac{\mathrm{d}^k(\alpha_1 y_1 + \alpha_2 y_2)}{\mathrm{d}x^k} + \cdots + a_1(x)\frac{\mathrm{d}(\alpha_1 y_1 + \alpha_2 y_2)}{\mathrm{d}x} + a_0(x)(\alpha_1 y_1 + \alpha_2 y_2) = 0,$$

which shows that $\alpha_1 y_1 + \alpha_2 y_2$ is also a solution of the DE. ■

Remark 31 *The fact that all the above holds relies on nothing more than the rules*

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}, \qquad \frac{d}{dx}(\alpha f) = \alpha\frac{df}{dx}$$

for functions f, g and real numbers α. These rules say that differentiation is a ***linear map****.*

In the case when the DE is linear, but inhomogeneous, solving the inhomogeneous equation still strongly relates to the solution of the associated homogeneous equation.

Theorem 32 *Let $Y(x)$ be a solution, known as a* ***particular solution****, or* ***particular integral****, of the inhomogeneous linear DE*

$$a_k(x)\frac{d^k y}{dx^k} + a_{k-1}(x)\frac{d^{k-1} y}{dx^{k-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)\,y = f(x). \qquad (3.9)$$

That is $y = Y$ satisfies the above. Then a function $y(x)$ is a solution of the inhomogeneous linear DE (3.9) if and only if $y(x)$ can be written as

$$y(x) = z(x) + Y(x)$$

where $z(x)$ is a solution of the corresponding homogeneous linear DE

$$a_k(x)\frac{d^k z}{dx^k} + a_{k-1}(x)\frac{d^{k-1} z}{dx^{k-1}} + \cdots + a_1(x)\frac{dz}{dx} + a_0(x)\,z = 0. \qquad (3.10)$$

The solution $z(x)$ to the corresponding homogeneous DE is known as the ***complementary function****.*

Proof. If $y(x) = z(x) + Y(x)$ is solution of (3.9) then

$$a_k(x)\frac{\mathrm{d}^k(Y+z)}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}(Y+z)}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}(Y+z)}{\mathrm{d}x} + a_0(x)(Y+z) = f(x).$$

Rearranging the brackets we get

$$\left(a_k(x)\frac{\mathrm{d}^k z}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}z}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}z}{\mathrm{d}x} + a_0(x)z\right) + \left(a_k(x)\frac{\mathrm{d}^k Y}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}Y}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}Y}{\mathrm{d}x} + a_0(x)Y\right) = f(x).$$

Now the second bracket equals $f(x)$ as $Y(x)$ is a particular solution of (3.9). Hence the first bracket must equal zero — that is $z(x)$ is a solution of the corresponding homogeneous DE (3.10). ■

Remark 33 *In practice, a particular solution is usually found by educated guess work and trial and error with functions that are roughly of the same type as* $f(x)$. *There are techniques to find particular solutions in the general case but you will not see these until next year.*

Remark 34 *The space of solutions of (3.9) is not a vector space. They form what is known as an* ***affine space***. *A homogeneous linear DE always has* 0 *as a solution, whereas this is not the case for the inhomogeneous equation. Compare this with 3D geometry: a plane through the origin is a vector space and if vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *span it then every point will have a position vector* $\lambda\mathbf{a} + \mu\mathbf{b}$; *points on a plane parallel to it will have position vectors* $\mathbf{p} + \lambda\mathbf{a} + \mu\mathbf{b}$ *where* $\mathbf{p}$ *is some point on the plane. The point* $\mathbf{p}$ *acts as a choice of origin in the plane, playing the same role as* Y *in the above.*

Example 35 *Find the general solution of*

$$x\frac{d^2y}{dx^2} + 2(1-x)\frac{dy}{dx} - 2y = 12x. \tag{3.11}$$

Solution. We already showed in Example 28 that the general solution of the corresponding homogeneous equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(1-x)\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$$

is

$$y(x) = \frac{ae^{2x} + b}{x}.$$

So we just need to find a particular solution of (3.11). A reasonable first attempt would be to see if there is a solution of the form

Such a Y is a solution if

So we see that $A = -3$ and $B = -3$. That is $Y(x) = -3x - 3$ is a particular solution. So the general solution of (3.11) is

$$y(x) = \frac{ae^{2x} + b}{x} - 3x - 3.$$

■

We will meet further examples of inhomogeneous linear DEs in Section 4.2.

4. LINEAR ODES WITH CONSTANT COEFFICIENTS

For the next two lectures we will look to treat the theory of solving linear differential equations

$$a_k \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1} \frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x)$$

where the functions $a_0, a_1, \ldots, a_k$ are **constants**.

We have already seen in Theorem 32 that the difference between solving the inhomogeneous and homogeneous is in finding a particular solution, so for now we will concentrate on the homogeneous case.

Mainly we shall treat examples when the equations are second order, though the theory extends naturally to similar higher order equations. We begin with the example of **simple harmonic motion (SHM).** This is the equation describing the vibrating of a spring or the swinging of a pendulum through small oscillations. The DE governing such motions is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\omega^2 y. \tag{4.1}$$

Example 36 *Show that the general solution of (4.1) is of the form*

$$y(x) = A\cos\omega x + B\sin\omega x.$$

The constant ω is the angular frequency of these oscillations, with the solutions having period $2\pi/\omega$.

Solution. We firstly set $v = \mathrm{d}y/\mathrm{d}x$. By the chain rule

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}y} = v\frac{\mathrm{d}v}{\mathrm{d}y}$$

and the differential equation (4.1) becomes

$$v\frac{\mathrm{d}v}{\mathrm{d}y} = -\omega^2 y$$

which is a separable one. If we separate the variables and integrate we find

$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 y^2 + K.$$

Recalling $v = \mathrm{d}y/\mathrm{d}x$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{K - \omega^2 y^2}.$$

Again this is separable and we may solve this to find

$$x = \int \frac{\mathrm{d}y}{\sqrt{K - \omega^2 y^2}}.$$

We met such integrals in the very first section on Trigonometric Substitutions, a sensible one here being

$$y = \frac{\sqrt{K}}{\omega}\sin t$$

which simplifies the integral to

$$x = \int \frac{\left(\sqrt{K}/\omega\right)\cos t}{\sqrt{K}\cos t}\,\mathrm{d}t = \frac{t}{\omega} + L,$$

for some constant L. Recalling $y = \left(\sqrt{K}/\omega\right)\sin t$ we have

$$y = \frac{\sqrt{K}}{\omega}\sin\omega\left(x - L\right)$$

or alternatively

$$y = A\cos\omega x + B\sin\omega x$$

for different constants A and B, by using the $\sin\left(\alpha + \beta\right)$ formula. ■

4.1 The Homogeneous Case

The SHM equation is a special case of the more general DE we are interested in, which is treated by the following theorem.

Theorem 37 *Consider the DE*

$$\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0 \tag{4.2}$$

with ***auxiliary equation*** *(AE)*

$$m^2 + Qm + R = 0.$$

The general solution to the DE is:

1. *in the case when the AE has two distinct real solutions α and β:*
$$Ae^{\alpha x} + Be^{\beta x};$$
2. *in the case when the AE has a repeated real solution α:*
$$(Ax + B)\, e^{\alpha x};$$
3. *in the case when the AE has complex conjugate roots $\alpha + i\beta$ and $\alpha - i\beta$:*
$$e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right).$$

Note also that the following proof is not typically examined and only a knowledge of the above solutions is expected.

Note, in the previous SHM example, we have tackled the case when the auxiliary equation's roots are $\pm\omega i$. I have aimed in the following to present proofs that will suit both those who are au fait with complex numbers and those who are not. Those with no or limited knowledge of complex numbers need only to recognise that quadratic equations which don't have real roots may be written in the form $(x-\alpha)^2 + \beta^2 = 0$ for some real α and β. For those who have not met complex numbers, or who have only done so in a limited way, there is a printout available on the course website. For those who have met Euler's result that

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

the form of the solution in case three will be less surprising.

Proof. Cases 1 and 2: Let's call the roots of the AE α and β, and presume for the moment that they are real roots, but not necessarily distinct. We can rewrite the original DE (4.2) as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (\alpha+\beta)\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha\beta y = 0.$$

Firstly note that $Y(x) = e^{\beta x}$ is a solution as substituting this in gives

$$\beta^2 e^{\beta x} - (\alpha + \beta)\,\beta e^{\beta x} + \alpha\beta e^{\beta x} = 0.$$

We saw in Section 3.2 how knowledge of a solution can simplify a second order homogeneous linear DE — if we set $z(x) = y(x)\,e^{-\beta x}$ then from equation (3.6) we have

$$e^{\beta x}\frac{\mathrm{d}w}{\mathrm{d}x} + \left(2\beta e^{\beta x} + (-\alpha - \beta)\,e^{\beta x}\right) w = 0$$

where $w = \mathrm{d}z/\mathrm{d}x$. Rearranging somewhat we find

$$\frac{\mathrm{d}w}{\mathrm{d}x} = (\alpha - \beta)\,w. \tag{4.3}$$

We now have two cases to consider: when $\alpha = \beta$ and when $\alpha \neq \beta$. In the case when the roots are equal then (4.3) leads to the following line of argument

$$\begin{aligned} w(x) &= \mathrm{d}z/\mathrm{d}x = A \quad \text{(a constant)}, \\ z(x) &= Ax + B \quad (A \text{ and } B \text{ constants}), \\ y(x) &= z(x)\,e^{\beta x} = (Ax + B)\,e^{\beta x}, \end{aligned}$$

as we stated in Case 2 of the theorem.

In the case when the roots are distinct reals then (4.3) has solution from Example 16

$$w(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = c_1 e^{(\alpha - \beta)x}$$

(where c_1 is a constant) and so integrating gives

$$z(x) = \frac{c_1}{\alpha - \beta} e^{(\alpha - \beta)x} + c_2$$

(where c_2 is a second constant) to finally find

$$y(x) = z(x)\,e^{\beta x} = \frac{c_1}{\alpha - \beta} e^{\alpha x} + c_2 e^{\beta x}.$$

Case 3: Suppose now that the roots of the equation are conjugate complex numbers $\alpha \pm i\beta$. For those who are aware of Euler's relation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

then we can simply treat this case the same as Case 1. Allowing for A and B to be complex numbers then (4.2) has a general solution of the form

$$y(x) = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} = e^{\alpha}\left(A'\cos\beta x + B'\sin\beta x\right).$$

Alternatively here is a proof which does not rely on complex numbers or Euler's relation. The original DE (4.2) when the AE has roots $\alpha \pm i\beta$ is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\alpha\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\alpha^2+\beta^2\right)y = 0. \tag{4.4}$$

We will first make the substitution

$$z(x) = y(x)\,e^{-\alpha x}.$$

Though it is not the case that $e^{\alpha x}$ is a solution this substitution will transform the DE into something familiar. Note

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{\alpha x}\right) = \frac{\mathrm{d}z}{\mathrm{d}x}e^{\alpha x} + \alpha z e^{\alpha x}, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\alpha x} + 2\alpha\frac{\mathrm{d}z}{\mathrm{d}x}e^{\alpha x} + \alpha^2 z e^{\alpha x}.$$

Hence (4.4) has become a new DE involving $z(x)$

$$\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\alpha x} + 2\alpha\frac{\mathrm{d}z}{\mathrm{d}x}e^{\alpha x} + \alpha^2 z e^{\alpha x}\right) - 2\alpha\left(\frac{\mathrm{d}z}{\mathrm{d}x}e^{\alpha x} + \alpha z e^{\alpha x}\right) + \left(\alpha^2+\beta^2\right)z e^{\alpha x} = 0,$$

which simplifies to

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\alpha x} + \beta^2 z e^{\alpha x} = 0.$$

Dividing through by $e^{\alpha x}$ gives

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = -\beta^2 z$$

which we recognise as the DE for SHM. This has general solution $z = A\cos\beta x + B\sin\beta x$ and so we can conclude

$$y(x) = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$$

as required. ■

Example 38 *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Solution.

■

Example 39 *Solve the equation*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0,$$

with initial conditions

$$y\,(0) = 1, \quad y'\,(0) = 0.$$

Solution. This has auxiliary equation

$$0 = m^2 - 3m + 2 = (m-1)\,(m-2)$$

which has roots $m = 1$ and $m = 2$. So the general solution of the equation is

$$y\,(x) = Ae^x + Be^{2x}.$$

Now the initial conditions imply

$$\begin{aligned} 1 &= y\,(0) = A + B, \\ 0 &= y'\,(0) = A + 2B. \end{aligned}$$

Hence

$$A = 2 \quad \text{and} \quad B = -1.$$

So the *unique* solution of this DE with initial solutions is

$$y\,(x) = 2e^x - e^{2x}.$$

■

Example 40 *Find all solutions of the differential equation*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which satisfy the boundary conditions

$$y(0) = 0 \quad \textit{and} \quad y(\pi) = 0.$$

Solution. This has auxiliary equation

$$0 = m^2 - 2m + 2 = (m-1)^2 + 1$$

which has roots $m = 1 \pm i$. So the general solution of the equation is

$$y(x) = e^x (A\cos x + B\sin x).$$

Now the boundary conditions imply

$$\begin{aligned} 0 &= y(0) = A, \\ 0 &= y(\pi) = -e^{\pi}A. \end{aligned}$$

Hence $A = 0$ and there are no constraints on B. So the boundary-value problem has solutions of the form

$$y(x) = Be^x \sin x$$

for any B. ■

The theory behind the solving of homogeneous linear DEs with constant coefficients extends to all orders, not just to second order DEs, provided suitable adjustments are made.

Example 41 *Write down the general solution of the following DE*

$$\frac{d^7y}{dx^7} + \frac{d^6y}{dx^6} - \frac{d^5y}{dx^5} - 5\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = 0$$

Solution. This has auxiliary equation

With can see (with a little effort) that this factorises as

which has roots $1, -1+i$ and $-1-i$, all of which are repeated roots, the first three times. So the general solution of the DE is

$$y\left(x\right)=\left(Ax^2+Bx+C\right)e^x+\left(Dx+E\right)e^{-x}\cos x+\left(Fx+G\right)e^{-x}\sin x.$$

■

4.2 The Inhomogeneous Case

In the previous section we discussed homogeneous linear differential equations with constant coefficients — that is equations of the form

$$a_k\frac{\mathrm{d}^ky}{\mathrm{d}x^k}+a_{k-1}\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}}+\cdots+a_1\frac{\mathrm{d}y}{\mathrm{d}x}+a_0y=0.$$

These equations occur naturally, for example the simple harmonic motion equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\omega^2y=0$$

governing the oscillations of a spring freely vibrating. (Such an equation arises from Hooke's Law). However if the oscillations are being driven at another frequency Ω the equation could now look like

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\omega^2y=A\sin\Omega t,$$

which is an inhomogeneous linear DE with constant coefficients. As has already been noted in Theorem 32:

- **The solutions $y(x)$ of an inhomogeneous linear differential equation are of the form $z(x)+Y(x)$ where $z(x)$ is a complementary function, i.e. a solution of the corresponding homogeneous equation, and $Y(x)$ is a particular solution of the inhomogeneous equation.**

The particular solution $Y(x)$ is usually found by a mixture of educated guesswork and trial and error.

Example 42 *Find the general solution of*

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=x. \tag{4.5}$$

Solution. As the function on the right is $f(x)=x$ then it would seem sensible to *try* a function of the form

$$Y(x)=Ax+B,$$

where A and B are, as yet, undetermined constants. There is no presumption that such a solution exists, but this seems a sensible range of functions where we may well find a particular solution. Note that

$$\frac{\mathrm{d}Y}{\mathrm{d}x}=A \quad \text{and} \quad \frac{\mathrm{d}^2Y}{\mathrm{d}x^2}=0.$$

So *if* $Y(x)$ is a solution of (4.5) then substituting it in gives

$$0-3A+2(Ax+B)=x$$

and this is an equation which must hold for all values of x. So comparing the coefficients of x on both sides, and the constant coefficients,

$$\begin{aligned} 2A &= 1 \text{ giving } A=\frac{1}{2}, \\ -3A+2B &= 0 \text{ giving } B=\frac{3}{4}. \end{aligned}$$

What this means is that

$$Y(x)=\frac{x}{2}+\frac{3}{4}$$

is a particular solution of (4.5). Having already found the *complementary function,* that is the general solution of the corresponding homogeneous DE in Example 39 then by Theorem 32 we know the general solution of (4.5) is

$$y(x)=Ae^{x}+Be^{2x}+\frac{x}{2}+\frac{3}{4},$$

for constants A and B. ■

Example 43 *Solve the initial value problem*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}, \qquad y(0) = y'(0) = 1.$$

Solution. From Example 38 we know that the general solution of the corresponding homogeneous equation is

$$y = (Ax + B)\, e^{3x}.$$

This means that trying neither e^{3x} nor xe^{3x} as a particular solution would be worthwhile as substituting either of them into the LHS would both give 0.

Instead we will try a particular solution of the form $Y(x) = Ax^2e^{3x}$. For this Y we find

Hence a particular solution is $Y(x) = \frac{1}{2}e^{3x}$. The general solution of the given inhomogeneous DE is

$$y(x) = \left(\frac{x^2}{2} + Ax + B\right) e^{3x}.$$

Now $y(0) = B = 1$; as

$$y'(x) = \left(x + A + \frac{3x^2}{2} + 3Ax + 3B\right) e^{3x}$$

then $y'(0) = A + 3B = 1$ and so $A = -2$. Hence the initial value problem has solution

$$y(x) = \left(\frac{x^2}{2} - 2x + 1\right) e^{3x}.$$

■

Example 44 *Find particular solutions of the following DE*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = f(x)$$

where

- $f(x) = \sin x$ — *Simply trying* $Y(x) = A\sin x$ *would do no good as* $Y'(x)$ *would contain* $\cos x$ *terms whilst* $Y(x)$ *and* $Y''(x)$ *would contain* $\sin x$ *terms. Instead we need to try the more general*

- $f(x) = e^{3x}$ — *This causes few problems and, as we would expect, we can find a solution of the form* $Y(x) = Ae^{3x}$;
- $f(x) = e^{x}$ — *This is different to the previous case because we know* Ae^{x} *is part of the general solution to the corresponding homogeneous DE, and simply substituting in* $Y(x) = Ae^{x}$ *into the LHS will yield* 0. *Instead we can successfully try a solution of the form*

- $f(x) = xe^{2x}$ — *Again* Ae^{2x} *is part of the solution to the homogeneous DE. Also as with the previous function we can see that* Axe^{2x} *would only help us with a* e^{2x} *term on the RHS. So we need to 'move up' a further power and try a solution of the form*

- $f(x) = e^{x}\sin x$ — *Though this may look somewhat more complicated a particular solution of the form*

  can be found.

- $f(x) = \sin^2 x$ — *Making use of the identity* $\sin^2 x = (1 - \cos 2x)/2$ *we can see that there is a solution of the form*

5. SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

5.1 Some Facts relating to Matrices

We introduce here some basic definitions and properties relating to matrices. We will use matrices later to work with simultaneous differential equations like

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + 3y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3x + 4y,$$

though the theory extends more generally to n homogeneous linear differential equations with constant coefficients in n variables.

Definition 45 *A (real) $m \times n$* ***matrix*** *is an array of real numbers arranged into m rows and n columns. This array is typically placed inside brackets and we write*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

The notation a_{ij} denotes the ***entry*** *in the ith row and jth column.*

Example 46 *Let*

$$A = (a_{ij}) = \begin{pmatrix} 1 & -3 & \pi \\ e^2 & 2.5 & 0 \end{pmatrix}.$$

This is a 2×3 matrix — $a_{12} = -3$, and $a_{21} = e^2$.

- We shall only be interested in 2×2 matrices. There is a printout on matrices available on the course website.

Definition 47 *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

be two 2×2 matrices. These matrices can be ***added*** *to form the sum $A + B$ and* ***multiplied*** *to form the product AB as follows.*

$$\begin{aligned} A+B &= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}; \\ AB &= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}. \end{aligned}$$

Note, for example, that the 1st row, 2nd column entry in AB is calculated by taking the term by term products of the 1st row of A and 2nd column of B and adding them — that is

$$\begin{pmatrix} \boxed{a_{11}} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & \boxed{b_{12}} \\ b_{21} & \boxed{b_{22}} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & \boxed{a_{11}b_{12}+a_{12}b_{22}} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$$

If c is a real number, often referred to as a ***scalar*** *in this context then we can also form the* ***scalar multiple*** *cA by multiplying each entry of A by c, i.e.*

$$cA = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$$

Remark 48 *More generally it is possible to*

- *add an $m_1 \times n_1$ matrix to an $m_2 \times n_2$ matrix if $m_1 = m_2$ and $n_1 = n_2$ — the (i, j)th entry in the sum is the sum of the matrices' (i, j)th entries;*

- *multiply an $m_1 \times n_1$ matrix by an $m_2 \times n_2$ matrix if $n_1 = m_2$ to produce an $m_1 \times n_2$ matrix product — the (i, j)th entry in the product is calculated by taking the term by term products of the ith row of the first matrix and the jth column of the second and adding them;*

- *multiply any $m \times n$ matrix by any real number c to form a scalar multiple — the (i, j)th entry in the scalar multiple is c times the (i, j)th entry in the matrix.*

Example 49 *For the following 2×2 matrices A and B find $A + B, B + A, AB$ and BA. Let*

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix}.$$

Then

$$\begin{aligned}
A+B &= \begin{pmatrix} 2+0 & 1+3 \\ -1-2 & 3+5 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix}; \\
B+A &= \begin{pmatrix} 0+2 & 3+1 \\ -2-1 & 5+3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix}; \\
AB &= \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix} = \\
BA &= \begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0-3 & 0+9 \\ -4-5 & -2+15 \end{pmatrix} = \begin{pmatrix} -3 & 9 \\ -9 & 13 \end{pmatrix}. \\
3A &= \begin{pmatrix} 6 & 3 \\ -3 & 9 \end{pmatrix}; \qquad 3B = \begin{pmatrix} 0 & 9 \\ -6 & 15 \end{pmatrix} \\
3A+3B &= \begin{pmatrix} 6 & 12 \\ -9 & 24 \end{pmatrix} = 3\,(A+B)
\end{aligned}$$

- ***Note that*** $A+B=B+A$ ***and this is generally the case; however*** $AB \neq BA$ ***in this case and we see that matrix multiplication is not generally commutative.***
- ***On the other hand matrix multiplication is always associative. That is***

  $$(AB)\,C = A\,(BC)$$

  for any 2×2 ***matrices*** A, B ***and*** C*.(We shall not prove this here.)*
- ***The distributive law***

  $$aA + aB = a\,(A+B)$$

  holds for all matrices A, B ***and reals*** a***, as does the other distributive law***

  $$(a+b)\,A = aA + bA$$

  for all matrices A ***and reals*** a, b*.*

Definition 50 *The* ***trace*** *of a matrix, A, often written*

$\operatorname{tr}(A)$, *is the sum of the components along its leading diagonal. For a* 2×2 *matrix*

$$\operatorname{tr}(A) = \operatorname{tr}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} + a_{22}.$$

Definition 51 *The **determinant** of a* 2×2 *matrix* A *written* $\det A$ *or* $|A|$ *is*

$$\det A = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example 52 *With* A *and* B *as in the previous example we have*

$$\begin{aligned} \det A &= \det \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = 2 \times 3 - 1 \times (-1) = 7; \\ \det B &= \det \begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix} = 0 \times 5 - 3 \times (-2) = 6; \\ \det(A+B) &= \det \begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix} = 16 + 12 = 28 \neq \det A + \det B; \\ \det(AB) &= \det \begin{pmatrix} -2 & 11 \\ 6 & 12 \end{pmatrix} = -24 + 66 = 42 = \det A \det B. \end{aligned}$$

Proposition 53 *Let* A *and* B *be two* 2×2 *matrices. Then*

$$\det(AB) = \det A \times \det B.$$

Proof.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then

$$\begin{aligned} \det(AB) &= \det \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \\ &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= bgcf + aedh - bhce - afdg \\ &= (ad - bc)(eh - fg) \\ &= \det A \det B. \end{aligned}$$

■

Definition 54 *The* 2×2 ***identity*** *matrix is denoted by* I *(or* I_2 *if we wish to stress the* 2×2 *context) and equals*

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

It has the property that $IA=A=AI$ *for any* 2×2 *matrix* A*.*

Definition 55 *The* 2×2 ***zero*** *matrix is denoted by* 0 *(or* 0_2 *if we wish to stress the* 2×2 *context) and equals*

$$0=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

It has the property that $0A=0=A0$ *for any* 2×2 *matrix* A*.*

Definition 56 *Given a* 2×2 *matrix* A *then an* ***inverse*** *matrix for* A *(which may or may not exist) is a* 2×2 *matrix* B *such that*

$$AB=I=BA.$$

If A *has an inverse then it is unique (see below) and we denote it as* A^{-1}*.*

Note if A^{-1} *exists then*

$$1=\det I=\det\left(AA^{-1}\right)=\det A\det\left(A^{-1}\right)$$

means

$$\det A^{-1}=\frac{1}{\det A}.$$

If A *has an inverse then it is said to be* ***invertible****, and is said to be* ***singular*** *if it has no inverse.*

Proposition 57 *A* 2×2 *matrix* A *has an inverse if and only if* $\det A\neq 0$*. If* $\det A\neq 0$ *then the inverse equals*

$$A^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

where $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$*. If an inverse exists it is unique.*

Proof. If A has an inverse B then

$$1 = \det I = \det AB = \det A \det B$$

and so we see that $\det A \neq 0$. On the other hand if $\det A = ad - bc \neq 0$ then we see

$$\left\{ \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right\} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = I \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left\{ \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right\} = I.$$

So an inverse exists.

Let's suppose two inverses B and C for A existed. That is there were two matrices B and C such that

$$BA = I = AB \quad \text{and} \quad CA = I = AC.$$

As matrix multiplication is associative then

$$C = IC = (BA)\,C = B\,(AC) = BI = B.$$

Hence the inverse we have found is indeed unique. ■

Remark 58 *Two vectors* $\mathbf{v}, \mathbf{w}$ *are said to be* ***linearly dependent*** *or just* ***dependent*** *if they are scalar multiples of one another — i.e. if they are parallel — and are said to be* ***linearly independent*** *or just* ***independent*** *otherwise.*

Note that $ad = bc$ *if and only if* $\binom{a}{b}$ *and* $\binom{c}{d}$ *are parallel. So a* 2×2 *matrix is invertible if its columns (or equivalently rows) are independent vectors.*

Definition 59 *The* ***eigenvalues*** *of a* 2×2 *matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *are the roots of the quadratic equation*

$$\begin{aligned} \det(A - xI) &= \det \begin{pmatrix} a-x & b \\ c & d-x \end{pmatrix} \\ &= (a-x)(d-x) - bc \\ &= x^2 - (a+d)\,x + (ad-bc) \\ &= x^2 - \operatorname{tr}(A)x + \det(A) = 0. \end{aligned}$$

Note that ...

Example 60 *Find the eigenvalues of the following matrices*

$$A=\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}, \qquad B=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \qquad C=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution.

$$\begin{aligned}
\det(A-xI) &= \begin{vmatrix} 2-x & 1 \\ 6 & 3-x \end{vmatrix} = \\
\det(B-xI) &= \begin{vmatrix} 2-x & 1 \\ 0 & 2-x \end{vmatrix} = (x-2)^2\,; \\
\det(C-xI) &= \begin{vmatrix} -x & -1 \\ 1 & -x \end{vmatrix} = x^2+1.
\end{aligned}$$

So we see the eigenvalues are $0, 5$ for A, $2, 2$ for B, and $\pm i$ for C. ■

Definition 61 *Given a 2×2 matrix A and a 2×1 column vector $\mathbf{v}=\binom{x}{y}$ then we can form the product $A\mathbf{v}$ which is another 2×1 column vector as follows — essentially this is again matrix multiplication treating $\mathbf{v}$ as half of a 2×2 matrix. We define*

$$A\mathbf{v}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

*In this way A defines a map $\mathbb{R}^2\to\mathbb{R}^2$ known as a **linear map**. (cf. Michaelmas Linear Algebra course.) This means that*

$$A(c_1\mathbf{v}_1+c_2\mathbf{v}_2)=c_1A\mathbf{v}_1+c_2A\mathbf{v}_2$$

for vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and scalars c_1, c_2. In particular, $A\mathbf{0}=\mathbf{0}$.

Example 62 *Determine the following products.*

$$\begin{aligned}
\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 \\ 6 \end{pmatrix}; \\
\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \\
\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -4 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 2 \\ -4 \end{pmatrix}; \\
\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} &=
\end{aligned}$$

- Note that the matrix maps $\binom{1}{0}$ to its first column and $\binom{0}{1}$ to its second column; it is easy to check that this generally happens.
- Note, in the third line, that it is possible for a matrix to map a non-zero vector to the zero vector.
- Note that in the third and fourth lines the vectors are mapped to scalar multiples of themselves.

Proposition 63 *The vector equation* $A\mathbf{x} = \mathbf{0}$ *has a non-zero solution* $\mathbf{x} = \mathbf{v}$ *if and only if* $\det A = 0$.

Proof. If $\det A \neq 0$ then we know from Proposition 57 that A has an inverse A^{-1}. So if $A\mathbf{v} = \mathbf{0}$ then

$$\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\mathbf{0} = \mathbf{0},$$

and we see $\mathbf{0}$ is the only solution. On the other hand if $\det A = ad - bc = 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} ad - bc \\ 0 \end{pmatrix} = \mathbf{0}.$$

So $\binom{d}{-c}$ is a non-zero solution unless $c = d = 0$ in which case $\binom{-b}{a}$ will do, unless $a = b = c = d = 0$ in which case any non-zero vector will do! ■

Corollary 64 *If* $x = \lambda$ *is a real eigenvalue of* A, *so that* $\det(A - \lambda I) = 0$, *then there is a non-zero vector* $\mathbf{v}$ *such that* $(A - \lambda I)\,\mathbf{v} = \mathbf{0}$. *That is*

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Definition 65 *A non-zero vector* $\mathbf{v}$ *such that* $A\mathbf{v} = \lambda \mathbf{v}$ *for some scalar* λ *is called an* ***eigenvector*** *or a* λ***-eigenvector.*** λ *is the corresponding* ***eigenvalue.***

Note conversely if $A\mathbf{v} = \mu\mathbf{v}$ for some non-zero vector $\mathbf{v}$ then $(A - \mu I)\,\mathbf{x} = \mathbf{0}$ has a non-zero solution $\mathbf{v}$ and so by the previous proposition $\det(A - \mu I) = 0$; in particular μ is an eigenvalue of A.

Example 66 *Find all the eigenvectors of the matrices* A, B, C *from Example 60.*

Solution. A has eigenvalues 0 and 5. We need to solve the vector equations

$$\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}, \qquad \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}.$$

The first vector equation says that

$$2x + y = 0 \quad \text{and} \quad 6x + 3y = 0.$$

Note that the equations are really the same equation (being scalar multiples of one another) which is why they have non-zero solutions. We can set $x = c_1$ and then $y = -2c_1$. Working similarly with the second case we see that A's eigenvectors are of the form

B has repeated eigenvalue 2. We need to solve the vector equation

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

which has solutions

$$c\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note that B does not have two independent eigenvectors.

C has no real eigenvalues and so no real eigenvectors. ■

Definition 67 *A* 2×2 *matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is said to be* ***diagonal*** *if* $b = c = 0$*; that is it is of the form* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

Theorem 68 *Suppose that the* 2×2 *matrix* A *has distinct real eigenvalues* λ_1 *and* λ_2*. Let* $\mathbf{v}_i$ *for* $i = 1, 2$ *be corresponding* λ_i*-eigenvectors and let*

$$P = (\mathbf{v}_1 |\, \mathbf{v}_2),$$

that is P *has columns* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*. Then* P *is invertible and*

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Proof. Suppose that

$$P \begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{v}_1 |\, \mathbf{v}_2) \begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{0}. \tag{5.1}$$

Then

$$A(x\mathbf{v}_1 + y\mathbf{v}_2) = x\lambda_1\mathbf{v}_1 + y\lambda_2\mathbf{v}_2 = \mathbf{0}. \tag{5.2}$$

Now (5.2) minus λ_1 times (5.1) gives

$$(\lambda_2 - \lambda_1)\, y\mathbf{v}_2 = \mathbf{0}.$$

As $\lambda_1 \neq \lambda_2$ and $\mathbf{v}_2 \neq \mathbf{0}$, then $y = 0$ and hence $x = 0$. As (5.1) only has $\mathbf{0}$ as a solution then P is invertible by Proposition 63.

Finally

$$AP = A(\mathbf{v}_1 |\, \mathbf{v}_2) = (\lambda_1\mathbf{v}_1 |\, \lambda_2\mathbf{v}_2) = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and premultiplying by P^{-1} on both sides gives the required result. ■

Example 69 *Let* A *be the matrix given in Example 60. Determine whether there is a real matrix* P*, such that* $P^{-1}AP$ *is diagonal. Are there similar matrices* P *for* B *and/or* C*?*

Solution. In Example 60 we showed that A had eigenvalues 0 and 5 with corresponding eigenvectors $\binom{1}{-2}$ and $\binom{1}{3}$ (see Example 66). From the previous theorem we know then there is such a P which has these two eigenvectors as its columns. We can check this directly works here:

$$\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 5 \\ 0 & 15 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$$

as required. A is said to be **diagonalisable**.

B on the other hand has eigenvalues 2 and 2. If such a P existed then it would follow that

$$P^{-1}BP = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I.$$

(This is the case as the eigenvalues of B and $P^{-1}BP$ would have to be the same ... Why?)

But then we'd have

$$B = P\,(2I)\,P^{-1} = 2PP^{-1} = 2I$$

and this is not the case; for B no such P exists and B is NOT diagonalisable.

Similarly for C, if such a real matrix P existed we'd have

$$P^{-1}CP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and this is clearly not possible with both P and C being real matrices. ■

5.2 Simultaneous Differential Equations

In this final section of the first course we shall be interested in coupled differential equations, governing two functions x and y of a variable t, when the equations have the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy, \qquad (a, b, c, d \in \mathbb{R})\,.$$

We already have two ways of approaching such systems, and we will meet a third way relating to the matrix theory we have recently met. We shall work with a specific example with initial values

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x + y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 6x + 4y, \qquad x\,(0) = y\,(0) = 1.$$

METHOD 1: If we use the first equation to write $y = \mathrm{d}x/\mathrm{d}t - 3x$, and substitute this into the second equation we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 3\frac{\mathrm{d}x}{\mathrm{d}t} = 6x + 4\frac{\mathrm{d}x}{\mathrm{d}t} - 12x,$$

and hence

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 7\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 0.$$

with $x(0) = 1$ and $x'(0) = 3x(0) + y(0) = 4$. We met in Chapter 4 how to solve such linear DEs with constant coefficients.

METHOD 2: Alternatively we could note

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3x + y}{6x + 4y}$$

and this is a homogeneous polar equation which we met previously in Section 2.4 and can be solved with a substitution of the form $y(x) = xv(x)$ to give an equation relating x and y. By substituting our expression for y back into the original equations in principle it should be possible to determine x and y in terms of t

METHOD 3: Our third method relates to the linear algebra we met in the previous section. We can rewrite our two equations as a single differential equation in a vector $\mathbf{v} = \binom{x}{y}$, namely

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} 3x + y \\ 6x + 4y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix} \mathbf{v}.$$

Let's write $A = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$. If we find A's eigenvalues, these turn out to be 1 and 6 and two corresponding independent eigenvectors are $\binom{1}{-2}$ and $\binom{1}{3}$ respectively. If we put these eigenvectors into the columns of a matrix

$$P = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \quad \text{so that} \quad P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix},$$

then we have

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = A\mathbf{v} = P \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} P^{-1}\mathbf{v}.$$

Hence, because the entries of P^{-1} are constant,

$$\frac{\mathrm{d}\left(P^{-1}\mathbf{v}\right)}{\mathrm{d}t}=\begin{pmatrix}1&0\\0&6\end{pmatrix}\left(P^{-1}\mathbf{v}\right).$$

If we set $\binom{X}{Y}=P^{-1}\mathbf{v}$ then we have

$$\frac{\mathrm{d}X}{\mathrm{d}t}=X\quad\text{and}\quad\frac{\mathrm{d}Y}{\mathrm{d}t}=6Y$$

with

$$\begin{pmatrix}X(0)\\Y(0)\end{pmatrix}=P^{-1}\begin{pmatrix}x(0)\\y(0)\end{pmatrix}=\frac{1}{5}\begin{pmatrix}3&-1\\2&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2/5\\3/5\end{pmatrix}.$$

Thus

$$X(t)=\frac{2}{5}e^{t}\quad\text{and}\quad Y(t)=\frac{3}{5}e^{6t}$$

and

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=P\begin{pmatrix}X(t)\\Y(t)\end{pmatrix}=\begin{pmatrix}1&1\\-2&3\end{pmatrix}\begin{pmatrix}\frac{2}{5}e^{t}\\\frac{3}{5}e^{6t}\end{pmatrix}=\begin{pmatrix}\frac{2}{5}e^{t}+\frac{3}{5}e^{6t}\\-\frac{4}{5}e^{t}-\frac{9}{5}e^{6t}\end{pmatrix}.$$

More generally for such systems of differential equations we have:

Theorem 70 *Two variables $x(t)$ and $y(t)$ satisfy the differential equations*

$$\frac{dx}{dt}=ax+by,\qquad\frac{dy}{dt}=cx+dy.\tag{5.3}$$

If the matrix $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ has distinct real eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ then the general solution of (5.3) is

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=Ae^{\lambda_1 t}\mathbf{v}_1+Be^{\lambda_2 t}\mathbf{v}_2.\tag{5.4}$$

Proof. The proof is simply a duplication of the working shown in Method 3 to the general case and so is omitted. ■

Whilst we won't go through the general details of what happens if the eigenvalues are repeated or complex we will treat two such examples below. Though it is worth noting

- **The formula (5.4) still holds when the eigenvalues λ_1 and λ_2 are complex provided we allow the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ to be complex and also the constants A and B. The solution can then be written as an overtly real function using the identity**

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Example 71 *Consider the linear system*

$$\frac{dx}{dt} = 2x + y, \qquad \frac{dy}{dt} = -4x + 6y, \qquad x(0) = y(0) = 1.$$

Solution. The matrix A of the system is

$$A = \begin{pmatrix} 2 & 1 \\ -4 & 6 \end{pmatrix}$$

and has repeated eigenvalue 4. We now make the substitutions

$$X(t) = x(t)\, e^{-4t}, \qquad Y(t) = y(t)\, e^{-4t}.$$

(More generally these would read $X(t) = x(t)\, e^{-\lambda t}$, etc. when λ is the repeated eigenvalue.) Then

$$\begin{aligned} \dot{X}(t) &= (\dot{x} - 4x)\, e^{-4t} = (-2x + y)\, e^{-4t} = -2X + Y, \\ \dot{Y}(t) &= (\dot{y} - 4y)\, e^{-4t} = (-4x + 2y)\, e^{-4t} = -4X + 2Y. \end{aligned}$$

We can see that $2\dot{X} - \dot{Y} = 0$ (generally $\dot{X}$ and $\dot{Y}$ will be proportional) and hence

$$2X - Y = \text{const.} = 2X(0) - Y(0) = 2 - 1 = 1.$$

Hence

$$\begin{aligned} \dot{X} &= -1 \text{ giving } X = -t + c_1, \\ \dot{Y} &= -2 \text{ giving } Y = -2t + c_2. \end{aligned}$$

As $X(0) = Y(0) = 1$ then $c_1 = c_2 = 1$ and we have our solution:

$$x(t) = (1 - t)\, e^{4t}, \qquad y(t) = (1 - 2t)\, e^{4t}.$$

■

Example 72 *Consider the linear system*

$$\frac{dx}{dt}=x+y, \qquad \frac{dy}{dt}=-x+y, \qquad x\left(0\right)=y\left(0\right)=1.$$

Solution. The matrix

$$A=\left(\begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array}\right)$$

has complex eigenvalues $1\pm i$. The corresponding eigenvectors are solutions of

$$\left(\begin{array}{cc} -i & 1 \\ -1 & -i \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right)=\mathbf{0}, \qquad \left(\begin{array}{cc} i & 1 \\ -1 & i \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right)=\mathbf{0},$$

and so we can take

$$\mathbf{v}_1=\left(\begin{array}{c} 1 \\ i \end{array}\right), \qquad \mathbf{v}_2=\left(\begin{array}{c} 1 \\ -i \end{array}\right).$$

The formula (5.4) holds still and we have

$$\left(\begin{array}{c} x\left(t\right) \\ y\left(t\right) \end{array}\right)=$$

As $x\left(0\right)=y\left(0\right)=1$ then

Hence

$$x\left(t\right)=\frac{\left(1-i\right)}{2}e^{\left(1+i\right)t}+\frac{\left(1+i\right)}{2}e^{\left(1-i\right)t}=\frac{e^t}{2}\left\{\left(e^{it}+e^{-it}\right)-i\left(e^{it}-e^{-it}\right)\right\}=e^t\left(\cos t+\sin t\right),$$

and similarly we find

$$y\left(t\right)=\frac{\left(i+1\right)}{2}e^{\left(1+i\right)t}+\frac{\left(-i+1\right)}{2}e^{\left(1-i\right)t}=\frac{e^t}{2}\left\{\left(e^{it}+e^{-it}\right)+i\left(e^{it}-e^{-it}\right)\right\}=e^t\left(\cos t-\sin t\right).$$

■

6. EPILOGUE — APPROXIMATIONS

We finish by looking at three examples where, perhaps in the face of not being able to solve the differential equations exactly, it is still possible to solve similar differential equations which approximate the more complicated original problem.

Example 73 *A mass P swinging on the end of a light rod is governed by the differential equation*

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

Suppose that the maximum swing of the mass is Θ. Show that the time T of an oscillation equals

$$T = 4\sqrt{\frac{l}{2g}}\int_0^\Theta \frac{d\theta}{\sqrt{\cos\theta - \cos\Theta}}$$

Solution. If we write $\omega = \mathrm{d}\theta/\mathrm{d}t$ then, using the chain rule, we have

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = -\frac{g}{l}\sin\theta.$$

Hence, separating variables and integrating, we see

$$\frac{1}{2}\omega^2 = \frac{g}{l}\cos\theta + C.$$

As $\omega = 0$ when $\theta = \Theta$, then $C = -\left(g/l\right)\cos\Theta$ and we can rearrange the previous equation to

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2g}{l}\left(\cos\theta - \cos\Theta\right)}$$

and hence

$$T = 4\sqrt{\frac{l}{2g}}\int_0^\Theta \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\Theta}}.$$

Normally of course we make the assumption that the pendulum moves through small oscillations and approximate $\sin\theta \approx \theta$ to get the standard SHM equation, whose solutions have a period of

$$T_A = 2\pi\sqrt{\frac{l}{g}}.$$

Graphically we can see how the two compare

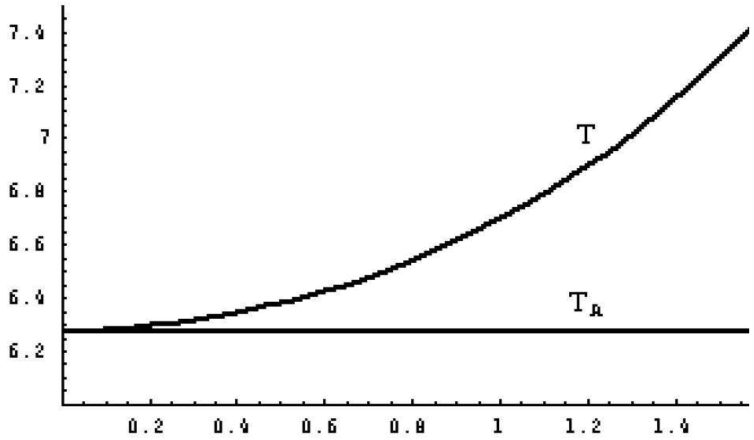


Figure 6-1 The approximation remains within a 1% error, that is $T/T_A < 1.01$, for $\theta < 0.398$ radians $\approx 22.8^\circ$, and within a 5% error, that is $T/T_A < 1.05$, for $\theta < 0.874$ radians $\approx 50^\circ$.

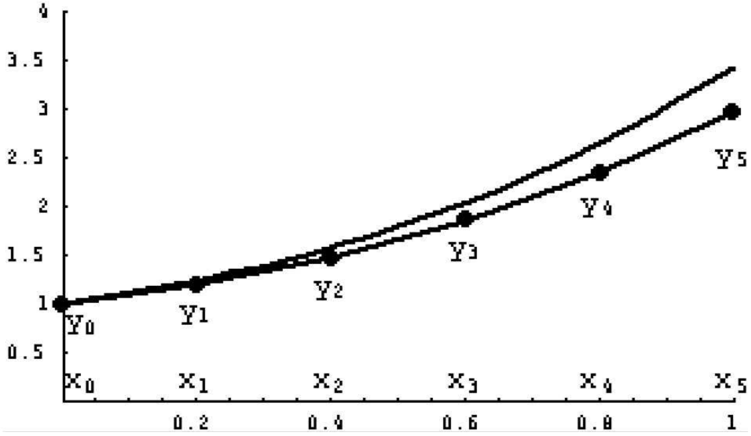
Definition 74 ***(Euler's Method)*** *This method suggests that an approximate solution to the initial-value problem*

$$\frac{dy}{dx} = f(x,y), \qquad y(a) = y_0$$

can be found using the iteration

$$y_{n+1} = y_n + hf(a+nh, y_n)$$

where $y_k = y(a+kh)$ and h is the length of increment in x. The method works on the assumption that y will continue to grow on the range $nh < x < (n+1)h$ at roughly the same rate as it was growing at $x = a+nh,\ y = y_n$. The smaller the h, the better the chance that this is a *reasonable assumption.*



Example 75 *Find, using Euler's method, an approximation to* $y(X)$ *where* y *satisfies the initial-value problem*

$$\frac{dy}{dx} = y + x, \qquad y(0) = 1.$$

Use an increment of X/N *and show that* $y_N \to y(X)$ *as* $N \to \infty$. *What value of* N *do we need to get* $y(1)$ *to within* 5%*? to within* 1%*?*

Solution. Using integrating factors we can solve the equation exactly to get

$$y(x) = 2e^x - 1 - x.$$

The iteration for Euler's method is

where $h = X/N$. Now, either by induction or with knowledge of difference equations, we can solve this recursion to get

Setting $h = X/N$ and $n = N$ we find

It is a well-known limit, from analysis, that $(1 + X/N)^N \to e^X$ as $N \to \infty$ and so y_N does indeed tend to $y(X)$ as $N \to \infty$.

To answer the second part, we note with $X = 1$ that

$$y_N = 2\left(1 + \frac{1}{N}\right)^N - 2, \text{ whilst } y(1) = 2e - 2.$$

With a calculator or computer we can find that

$$\begin{aligned} \frac{y_N}{y(1)} &> 0.95 \quad \text{for } N \geqslant 15; \\ \frac{y_N}{y(1)} &> 0.99 \quad \text{for } N \geqslant 79. \end{aligned}$$

■

Example 76 ***(Critical Points and Stability)*** *The* ***Volterra-Lotka*** *equations are*

$$\frac{dx}{dt} = -mx + axy, \qquad \frac{dy}{dt} = by - kxy,$$

where a, b, k, m are positive constants, and they model competing numbers of predators x and prey y. The ***critical points*** *(x, y) are points where the populations x and y remain constant, i.e. where $\dot{x} = \dot{y} = 0$. Solving these equations we see that the two critical points are*

$$(0,0) \quad \textit{and} \quad \left(\frac{b}{k}, \frac{m}{a}\right).$$

What would happen though if the steady populations of $x = b/k, y = m/a$ were disturbed a little, e.g. some few extra predators or prey introduced?

Solution. To see this we make a change of variable

$$X = x - \frac{b}{k}, \qquad Y = y - \frac{m}{a}.$$

The Volterra-Lotka equations then become

$$\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= -m\left(X + \frac{b}{k}\right) + a\left(X + \frac{b}{k}\right)\left(Y + \frac{m}{a}\right) = \frac{abY}{k} + aXY; \\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= b\left(Y + \frac{m}{a}\right) - k\left(X + \frac{b}{k}\right)\left(Y + \frac{m}{a}\right) = -\frac{kmX}{a} - kXY.
\end{aligned}$$

But X and Y represent only the small changes in populations, and so the second order term XY will be smaller still and negligible for small enough perturbations. So we can approximate the equations to

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \left(\frac{ab}{k}\right)Y, \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = \left(\frac{-km}{a}\right)X.$$

The equations

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \left(\frac{ab}{k}\right)Y, \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = \left(\frac{-km}{a}\right)X$$

form a linear system of equations like the ones we studied in the previous chapter. We could solve them using similar methods, but that is really unnecessary here as we can note

$$\begin{aligned}
\frac{\mathrm{d}^2X}{\mathrm{d}t^2} &= \left(\frac{ab}{k}\right)\frac{\mathrm{d}Y}{\mathrm{d}t} = \left(\frac{ab}{k}\right)\left(\frac{-km}{a}\right)X = -bmX, \\
\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} &= \left(\frac{-km}{a}\right)\frac{\mathrm{d}X}{\mathrm{d}t} = \left(\frac{-km}{a}\right)\left(\frac{ab}{k}\right)Y = -bmY,
\end{aligned}$$

which are SHM equations and X and Y are given by

$$X = \alpha_1 \cos\left(\sqrt{bm}t + \varepsilon_1\right), \qquad Y = \alpha_2 \cos\left(\sqrt{bm}t + \varepsilon_2\right), \tag{6.1}$$

for constants $\alpha_1, \alpha_2, \varepsilon_1, \varepsilon_2$. So the critical point $(b/k, m/a)$ is **stable**, meaning that if we perturb the populations from their steady states they remain roughly the same. In fact, from equations (6.1) we can see that the populations move around the critical point in a circular/elliptical motion. This type of critical point is called a **centre**. ■