Complex Numbers — An Introduction

Richard Earl

Michaelmas Term 2005

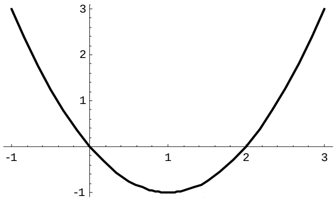
1 Their Algebra and Geometry

1.1 The Need For Complex Numbers

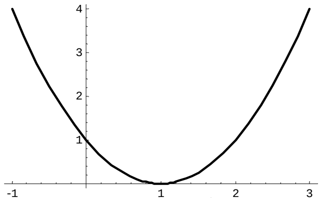
All of you will know that the two roots of the quadratic equation $ax^2+bx+c=0$ are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{1}$$

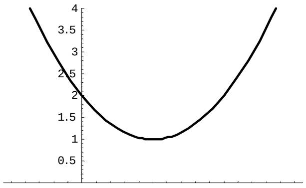
and solving quadratic equations is something that mathematicians have been able to do since the time of the Babylonians. When $b^2-4ac>0$ then these two roots are real and distinct; graphically they are where the curve $y=ax^2+bx+c$ cuts the x-axis. When $b^2-4ac=0$ then we have one real root and the curve just touches the x-axis here. But what happens when $b^2-4ac<0$? Then there are no real solutions to the equation as no real squares to give the negative b^2-4ac. From the graphical point of view the curve $y=ax^2+bx+c$ lies entirely above or below the x-axis.



Distinct real roots



Repeated real root



Complex roots

It is only comparatively recently that mathematicians have been comfortable with these roots when $b^2-4ac<0$. During the Renaissance the quadratic would have been considered unsolvable or its roots would have been called *imaginary* (a term first used by Descartes). If we imagine $\sqrt{-1}$ to exist, and that it behaves (adds and multiplies) much the same as other numbers then the two roots of the quadratic can be written in the form

$$x=A\pm B\sqrt{-1} \tag{2}$$

where

$$A=-\frac{b}{2a} \quad\text{and}\quad B=\frac{\sqrt{4ac-b^2}}{2a} \quad\text{are real numbers.}$$

But what meaning can such roots have? It was this philosophical point which pre-occupied mathematicians until the start of the 19th century when these 'imaginary' numbers started proving so useful (especially in the work of Cauchy and Gauss) that essentially the philosophical concerns just got forgotten about.

Notation 1 *We shall from now on write i for $\sqrt{-1}$, though many books, particularly those written for engineers and physicists use j instead. The notation i was first introduced by Euler.*

Definition 2 *A* ***complex number*** *is a number of the form $a+bi$ where a and b are real numbers. If $z=a+bi$ then a is known as the* ***real part*** *of z and b as the* ***imaginary part****. We write $a=\operatorname{Re}z$ and $b=\operatorname{Im}z$. Note that real numbers are complex — a real number is simply a complex number with no imaginary part.*

Notation 3 *We write $\mathbb{C}$ for the set of all complex numbers.*

One of the first major results concerning complex numbers and which conclusively demonstrated their usefulness was proved by Gauss in 1799. From the quadratic formula (1) we know that all quadratic equations can be solved using complex numbers — what Gauss was the first to prove was the much more general result:

Theorem 4 *(**Fundamental Theorem of Algebra**). The roots of any polynomial equation $a_0+a_1x+\cdots+a_nx^n=0$ with real (or complex) coefficients a_i are complex. That is there are n (not necessarily distinct) complex numbers $\gamma_1,\ldots,\gamma_n$ such that*

$$a_0+a_1x+a_2x^2+\cdots+a_nx^n=a_n\left(x-\gamma_1\right)\left(x-\gamma_2\right)\cdots\left(x-\gamma_n\right).$$

In particular the theorem shows that an n degree polynomial has, counting multiplicities, n roots in $\mathbb{C}$.

The proof of this theorem is far beyond the scope of this article. Note that the theorem only guarantees the *existence* of the roots of a polynomial somewhere in $\mathbb{C}$ unlike the quadratic formula which explicitly gives us the roots. The theorem gives no hints as to where in $\mathbb{C}$ these roots are to be found.

1.2 Basic Operations

We add, subtract, multiply and divide complex numbers much as we would expect. We add and subtract complex numbers by adding their real and imaginary parts:-

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)\,i,\\ (a+bi)-(c+di) &= (a-c)+(b-d)\,i.\end{aligned}$$

We can multiply complex numbers by expanding the brackets in the usual fashion and using $i^2=-1$,

$$(a+bi)\,(c+di)=ac+bci+adi+bdi^2=(ac-bd)+(ad+bc)\,i.$$

To divide complex numbers we note firstly that $(c+di)\,(c-di)=c^2+d^2$ is real. So

$$\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\times\frac{c-di}{c-di}=\left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i.$$

The number $c-di$ which we just used, as relating to $c+di$, has a special name and some useful properties — see Proposition 11.

Definition 5 *Let $z=a+bi$. The **conjugate** of z is the number $a-bi$ and this is denoted as $\bar{z}$ (or sometimes z^*).*

- Note from equation (2) that when the *real* quadratic equation $ax^2+bx+c=0$ has complex roots then these roots are conjugates of each other. Generally if z_0 is a root of the polynomial $a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0=0$ where the a_i are real then so is its conjugate $\overline{z_0}$.

Example 6 *Calculate, in the form $a+bi$, the following complex numbers:*

$$(1+3i)+(2-6i)\,,\quad(1+3i)-(2-6i)\,,\quad(1+3i)\,(2-6i)\,,\quad\frac{1+3i}{2-6i}.$$

Solution.

$$\begin{aligned}(1+3i)+(2-6i) &= (1+2)+(3+(-6))\,i=3-3i;\\ (1+3i)-(2-6i) &= (1-2)+(3-(-6))\,i=-1+9i.\\ (1+3i)\,(2-6i) &= 2+6i-6i-18i^2=2+18=20.\end{aligned}$$

Division takes a little more care, and we need to remember to multiply through by the conjugate of the denominator:

$$\frac{1+3i}{2-6i}=\frac{(1+3i)\,(2+6i)}{(2-6i)\,(2+6i)}=\frac{2+6i+6i+18i^2}{2^2+6^2}=\frac{-16+12i}{40}=\frac{-2}{5}+\frac{3}{10}i.$$

■

We present the following example because it is a common early misconception involving complex numbers — if we need a new number i as the square root of -1, then shouldn't we need another one for the square root of i? But $z^2=i$ is just another polynomial equation, with complex coefficients, and two (perhaps repeated) roots in $\mathbb{C}$ are guaranteed by the Fundamental Theorem of Algebra. They are also quite easy to calculate.

Example 7 *Find all those z that satisfy $z^2=i$.*

Solution. Suppose that $z^2 = i$ and $z = a + bi$, where a and b are real. Then

$$i = (a + bi)^2 = \left(a^2 - b^2\right) + 2abi.$$

Comparing the real and imaginary parts we see that

$$a^2 - b^2 = 0 \quad \text{and} \quad 2ab = 1.$$

So $b = \pm a$ from the first equation. Substituting $b = a$ into the second equation gives $a = b = 1/\sqrt{2}$ or $a = b = -1/\sqrt{2}$. Substituting $b = -a$ into the second equation of gives $-2a^2 = 1$ which has no real solution in a.

So the two z which satisfy $z^2 = i$, i.e. the two square roots of i, are

$$\frac{1+i}{\sqrt{2}} \quad \text{and} \quad \frac{-1-i}{\sqrt{2}}.$$

Notice, as with square roots of real numbers, that the two square are negative one another. ■

Example 8 *Use the quadratic formula to find the two solutions of*

$$z^2 - (3 + i)\, z + (2 + i) = 0.$$

Solution. We see that $a = 1$, $b = -3 - i$, and $c = 2 + i$. So

$$b^2 - 4ac = (-3 - i)^2 - 4 \times 1 \times (2 + i) = 9 - 1 + 6i - 8 - 4i = 2i.$$

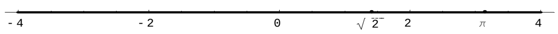
Knowing the square roots of i are $\pm (1 + i)/\sqrt{2}$ from the previous example, we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(3+i) \pm \sqrt{2i}}{2} = \frac{(3+i) \pm \sqrt{2}\sqrt{i}}{2} \\ &= \frac{(3+i) \pm (1+i)}{2} = \frac{4+2i}{2} \ \text{ or } \ \frac{2}{2} = 2 + i \ \text{ or } \ 1. \end{aligned}$$

Note that the two roots are not conjugates of one another — this need not be the case when the coefficients a, b, c are not all real. ■

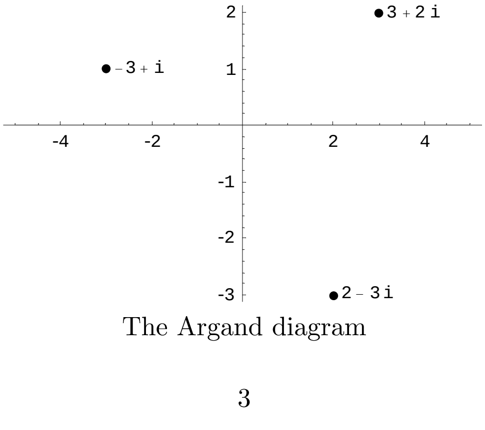
1.3 The Argand Diagram

The real numbers are often represented on the *real line* which increase as we move from left to right



The real number line

The complex numbers, having two components, their real and imaginary parts, can be represented as a plane; indeed $\mathbb{C}$ is sometimes referred to as the **complex plane**, but more commonly when we represent $\mathbb{C}$ in this manner we call it an **Argand diagram**. (after the Swiss mathematician Argand (1768-1822)). The point (a, b) represents the complex number $a + bi$ so that the x-axis contains all the real numbers, and so is termed the **real axis**, and the y-axis contains all those complex numbers which are purely imaginary (i.e. have no real part) and so is referred to as the **imaginary axis**.



The Argand diagram

Note that the conjugate $\bar{z}$ of a point z is its mirror image in the real axis. So, $z \mapsto \bar{z}$ represents reflection in the real axis.

A complex number z in the complex plane can be represented by Cartesian co-ordinates, its real and imaginary parts, but equally useful is the representation of z by polar co-ordinates. If we let r be the distance of z from the origin and, if $z \neq 0$, we let θ be the angle that the line connecting z to the origin makes with the positive real axis then we can write

$$z = x + iy = r\cos\theta + ir\sin\theta. \tag{3}$$

The relations between z's Cartesian and polar co-ordinates are simple — we see that

$$\begin{aligned} x &= r\cos\theta \text{ and } y = r\sin\theta, \\ r &= \sqrt{x^2+y^2} \text{ and } \tan\theta = \frac{y}{x}. \end{aligned}$$

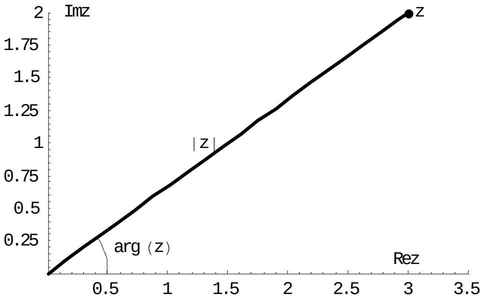
Definition 9 *The number r is called the **modulus** of z and is written $|z|$. If $z = x+iy$ then*

$$|z| = \sqrt{x^2+y^2}.$$

Definition 10 *The number θ is called the **argument** of z and is written $\arg z$. If $z = x + iy$ then*

$$\sin\arg z = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\arg z = \frac{x}{\sqrt{x^2+y^2}} \quad \textit{and} \quad \tan\arg z = \frac{y}{x}.$$

*Note that the argument of 0 is undefined. Note also that $\arg z$ is defined only up to multiples of 2π. For example, the argument of $1+i$ could be $\pi/4$ or $9\pi/4$ or $-7\pi/4$ etc.. **For simplicity we shall give all arguments in the range** $0 \leq \theta < 2\pi$, so that $\pi/4$ would be the preferred choice here.*



A Complex Number's Cartesian and Polar Co-ordinates

We now prove some important formulae about properties of the modulus, argument and conjugation.

Proposition 11 *The modulus, argument and conjugate functions satisfy the following properties. Let $z, w \in \mathbb{C}$. Then*

$$\begin{aligned}
|zw| &= |z|\,|w|, & (4)\\
\left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \textit{ if } w \neq 0, & (5)\\
\overline{z \pm w} &= \bar{z} \pm \overline{w}, & (6)\\
\overline{zw} &= \bar{z}\,\overline{w}, & (7)\\
\arg(zw) &= \arg z + \arg w \textit{ if } z, w \neq 0, & (8)\\
z\bar{z} &= |z|^2, & (9)\\
\arg\left(\frac{z}{w}\right) &= \arg z - \arg w \quad \textit{if} \quad z, w \neq 0, & (10)\\
\overline{\left(\frac{z}{w}\right)} &= \frac{\bar{z}}{\overline{w}} \quad \textit{if} \quad w \neq 0, & (11)\\
|\bar{z}| &= |z|, & (12)\\
\arg\bar{z} &= -\arg z, & (13)\\
|z+w| &\leq |z| + |w|, & (14)\\
||z| - |w|| &\leq |z - w|. & (15)
\end{aligned}$$

Proof. Identity (4): $|zw| = |z|\,|w|$.
Let $z = a + bi$ and $w = c + di$. Then $zw = (ac - bd) + (bc + ad)\,i$ so that

$$\begin{aligned} |zw| &= \sqrt{(ac-bd)^2 + (bc+ad)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} \\ &= \sqrt{(a^2+b^2)(c^2+d^2)} \\ &= \sqrt{a^2+b^2}\sqrt{c^2+d^2} = |z|\,|w|. \end{aligned}$$

■

Proof. Identity (8): $\arg(zw) = \arg z + \arg w$.
Let $z = r(\cos\theta + i\sin\theta)$ and $w = R(\cos\Theta + i\sin\Theta)$. Then

$$\begin{aligned} zw &= rR(\cos\theta + i\sin\theta)(\cos\Theta + i\sin\Theta) \\ &= rR((\cos\theta\cos\Theta - \sin\theta\sin\Theta) + i(\sin\theta\cos\Theta + \cos\theta\sin\Theta)) \\ &= rR(\cos(\theta+\Theta) + i\sin(\theta+\Theta)). \end{aligned}$$

We can read off that $|zw| = rR = |z|\,|w|$, which is a second proof of the previous part, and also that

$$\arg(zw) = \theta + \Theta = \arg z + \arg w, \quad \text{up to multiples of } 2\pi.$$

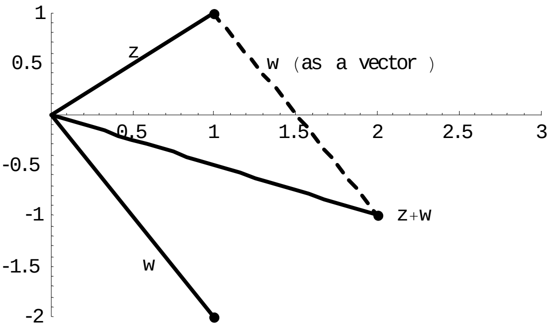
■

Proof. Identity (7): $\overline{zw} = \bar{z}\,\bar{w}$.
Let $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned} \overline{zw} &= \overline{(ac-bd) + (bc+ad)\,i} \\ &= (ac-bd) - (bc+ad)\,i \\ &= (a-bi)(c-di) = \bar{z}\,\bar{w}. \end{aligned}$$

■

Proof. Identity (14): the Triangle Inequality $|z + w| \leq |z| + |w|$. A diagrammatic proof of this is simple and explains the inequality's name:



A Diagrammatic Proof Of The Triangle Inequality

Note that the shortest distance between 0 and $z + w$ is the modulus of $z + w$. This is shorter in length than the path which goes from 0 to z to $z + w$. The total length of this second path is $|z| + |w|$. For an algebraic proof, note that for any complex number

$$z + \bar{z} = 2\operatorname{Re} z \quad \text{and} \quad \operatorname{Re} z \leq |z|.$$

So for $z, w \in \mathbb{C}$,

$$\frac{z\bar{w} + \bar{z}w}{2} = \operatorname{Re}(z\bar{w}) \leq |z\bar{w}| = |z|\,|\bar{w}| = |z|\,|w|.$$

Then

$$\begin{aligned} |z+w|^2 &= (z+w)\overline{(z+w)} \\ &= (z+w)(\bar{z}+\bar{w}) \\ &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\ &\leq |z|^2 + 2|z|\,|w| + |w|^2 = (|z| + |w|)^2, \end{aligned}$$

to give the required result. ■

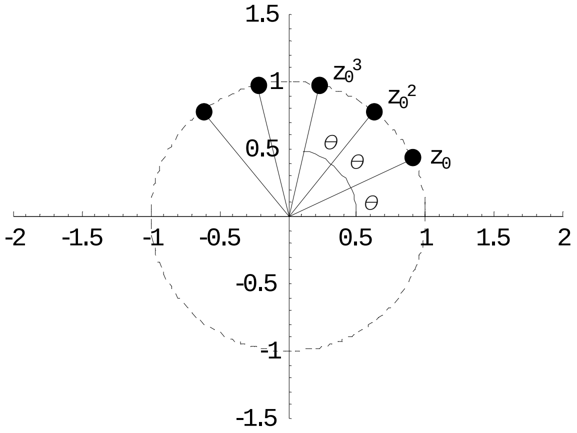
The remaining identities are left to Exercise 7

1.4 Roots Of Unity

Consider the complex number

$$z_0 = \cos\theta + i\sin\theta$$

where θ is some real number in the range $0 \leqslant \theta < 2\pi$. The modulus of z_0 is 1 and the argument of z_0 is θ.



Powers of z_0

In Proposition 11 we proved for $z, w \neq 0$ that

$$|zw| = |z|\,|w| \quad \text{and} \quad \arg(zw) = \arg z + \arg w.$$

So for any integer n, and any $z \neq 0$, we have that

$$|z^n| = |z|^n \quad \text{and} \quad \arg(z^n) = n\arg z.$$

Then the modulus of $(z_0)^n$ is 1, and the argument of $(z_0)^n$ is $n\theta$ up to multiples of 2π. Putting this another way, we have the famous theorem due to De Moivre:

Theorem 12 ***(De Moivre's Theorem)*** *For a real number θ and integer n we have that*

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

We apply these ideas now to the following:

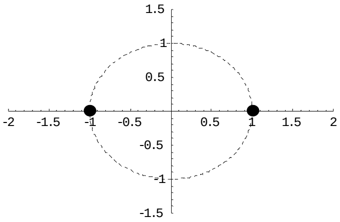
Example 13 *Let $n \geqslant 1$ be a natural number. Find all those complex z such that $z^n = 1$.*

Solution. We know from the Fundamental Theorem of Algebra that there are (counting multiplicities) n solutions — these are known as *the nth roots of unity.* Let's first solve $z^n = 1$ directly for $n = 2, 3, 4$.

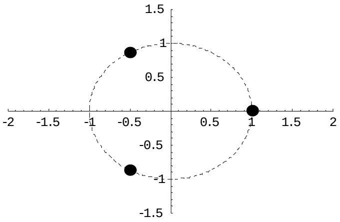
- When $n = 2$ we have
$$0 = z^2 - 1 = (z-1)(z+1)$$
and so the square roots of 1 are ±1.
- When $n = 3$ we can factorise as follows
$$0 = z^3 - 1 = (z-1)\left(z^2 + z + 1\right).$$
So 1 is a root and completing the square we see
$$0 = z^2 + z + 1 = \left(z + \frac{1}{2}\right)^2 + \frac{3}{4}$$
which has roots $-1/2 \pm \sqrt{3}i/2$. So the cube roots of 1 are
$$1 \text{ and } \frac{-1}{2} + \frac{\sqrt{3}}{2}i \text{ and } \frac{-1}{2} - \frac{\sqrt{3}}{2}i.$$

- When $n=4$ we can factorise as follows
$$0=z^4-1=\left(z^2-1\right)\left(z^2+1\right)=(z-1)(z+1)(z-i)(z+i),$$
so that the fourth roots of 1 are $1,-1,i$ and $-i$.

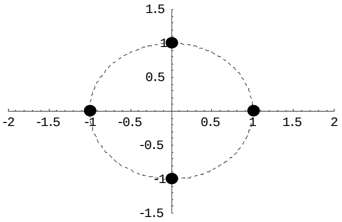
Plotting these roots on Argand diagrams we can see a pattern developing



Square Roots



Cube Roots



Fourth Roots

Returning to the general case suppose that
$$z=r(\cos\theta+i\sin\theta)\quad\text{and satisfies}\quad z^n=1.$$
Then by the observations preceding De Moivre's Theorem z^n has modulus r^n and has argument $n\theta$ whilst 1 has modulus 1 and argument 0. Then comparing their moduli
$$r^n=1\implies r=1.$$
Comparing arguments we see $n\theta=0$ up to multiples of 2π. That is $n\theta=2k\pi$ for some integer k, giving $\theta=2k\pi/n$. So we see that if $z^n=1$ then z has the form
$$z=\cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right)\quad\text{where } k \text{ is an integer.}$$
At first glance there seems to be an infinite number of roots but, as cos and sin have period 2π, then these z repeat with period n. ■

Hence we have shown

Proposition 14 *The nth roots of unity, that is the solutions of the equation $z^n=1$, are*
$$z=\cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right)\quad\textit{where}\quad k=0,1,2,\ldots,n-1.$$
Plotted on an Argand diagram these nth roots of unity form a regular n-gon inscribed within the unit circle with a vertex at 1.

Example 15 *Find all the solutions of the cubic $z^3=-2+2i$.*

Solution. If we write $-2+2i$ in its polar form we have
$$-2+2i=\sqrt{8}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right).$$
So if $z^3=-2+2i$ and z has modulus r and argument θ then
$$r^3=\sqrt{8}\quad\text{and}\quad 3\theta=\frac{3\pi}{4}\quad\text{up to multiples of } 2\pi,$$
which gives
$$r=\sqrt{2}\quad\text{and}\quad\theta=\frac{\pi}{4}+\frac{2k\pi}{3}\quad\text{for some integer } k.$$
As before we need only consider $k=0,1,2$ (as other values of k lead to repeats) and so the three roots are
$$\begin{aligned}
\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right) &= 1+i,\\
\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right) &= \left(\frac{-1}{2}-\frac{\sqrt{3}}{2}\right)+i\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right),\\
\sqrt{2}\left(\cos\left(\frac{19\pi}{12}\right)+i\sin\left(\frac{19\pi}{12}\right)\right) &= \left(\frac{-1}{2}+\frac{\sqrt{3}}{2}\right)+i\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\right).
\end{aligned}$$

■

1.5 Distance and Angles in the Complex Plane

Let $z = z_1 + iz_2$ and $w = w_1 + iw_2$ be two complex numbers. By Pythagoras' Theorem the distance between z and w as points in the complex plane equals

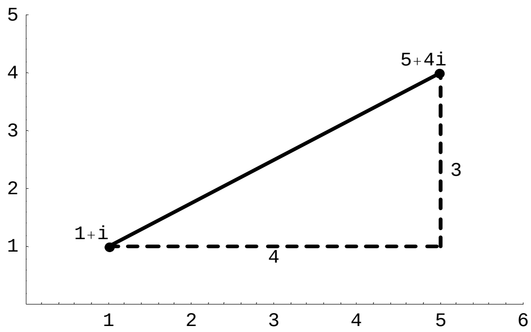
$$\begin{aligned}\text{distance} &= \sqrt{(z_1 - w_1)^2 + (z_2 - w_2)^2} \\ &= |(z_1 - w_1) + i(z_2 - w_2)| \\ &= |(z_1 + iz_2) - (w_1 + iw_2)| \\ &= |z - w|.\end{aligned}$$

Let $a = a_1 + ia_2$, $b = b_1 + ib_2$, and $c = c_1 + ic_2$ be three points in the complex plane representing three points A, B and C. To calculate the angle $\measuredangle BAC$, as in the second diagram below where we measure anti-clockwise from AB to AC, we find

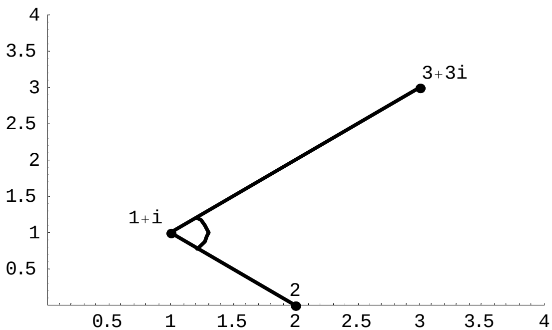
$$\measuredangle BAC = \arg(c - a) - \arg(b - a) = \arg\left(\frac{c - a}{b - a}\right).$$

Note that if in the formula B and C were switched then we get the larger angle

$$\arg\left(\frac{b - a}{c - a}\right) = 2\pi - \measuredangle BAC.$$



The distance here is $\sqrt{3^2 + 4^2} = 5$



The angle is $\arg\left(\frac{2+2i}{1-i}\right) = \arg 2i = \frac{1}{2}\pi$

Example 16 *Find the smaller angle* $\measuredangle BAC$ *where* $a = 1 + i$, $b = 3 + 2i$, *and* $c = 4 - 3i$.

Solution. The angle $\measuredangle BAC$ is given by

$$\arg\left(\frac{b - a}{c - a}\right) = \arg\left(\frac{2 + i}{3 - 4i}\right) = \arg\left(\frac{2 + 11i}{25}\right) = \tan^{-1}\left(\frac{11}{2}\right).$$

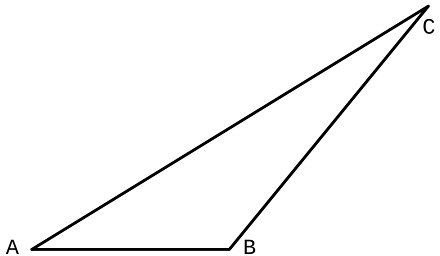
■

Example 17 *Prove the cosine rule for triangles using complex geometry. Recall that the cosine rule states: Let ABC be a triangle. Then*

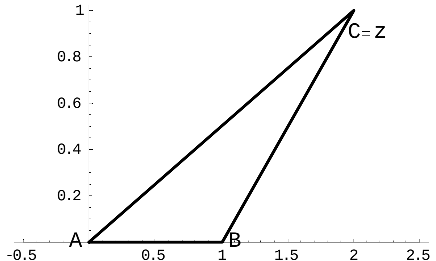
$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos\hat{A}. \tag{16}$$

Solution. We can choose our co-ordinates in the plane so that A is at the origin and B is at 1. Let C be at the point z. So in terms of our co-ordinates:

$$|AB| = 1, \qquad |BC| = z - 1, \qquad |AC| = |z|, \qquad \hat{A} = \arg z.$$



Triangle in featureless plane



With co-ordinates added

■

Proof. So

$$\begin{aligned}
\text{RHS of (16)} &= |z|^2 + 1 - 2|z|\cos\arg z \\
&= z\overline{z} + 1 - 2|z| \times \frac{\operatorname{Re} z}{|z|} \\
&= z\overline{z} + 1 - 2 \times \frac{(z+\overline{z})}{2} \\
&= z\overline{z} + 1 - z - \overline{z} \\
&= (z-1)(\overline{z}-1) \\
&= |z-1|^2 = \text{LHS of (16)}.
\end{aligned}$$

■

1.6 Transformations of the Complex Plane

We now describe some transformations of the complex plane and show how they can be written in terms of complex numbers.

- **Translations:** A translation of the plane is one which takes the point (x, y) to the point $(x + a, y + b)$ where a and b are two real constants. In terms of complex co-ordinates this is the map $z \mapsto z + z_0$ where $z_0 = a + ib$.
- **Rotations:** Consider rotating the plane about the origin anti-clockwise through an angle α. If we take an arbitrary point in polar form $r(\cos\theta + i\sin\theta)$ then this will rotate to the point

  $$r(\cos(\theta+\alpha) + i\sin(\theta+\alpha)) = r(\cos\theta + i\sin\theta)(\cos\alpha + i\sin\alpha)$$

  as the argument of a product is the sum of the arguments. Hence rotation about the origin by α anticlockwise, is represented in complex co-ordinates as the map

  $$z \mapsto z(\cos\alpha + i\sin\alpha).$$

  More generally, any rotation of $\mathbb{C}$, not necessarily about the origin, has the form $z \mapsto az + b$ where $a, b \in \mathbb{C}$, with $|a| = 1$ and $a \neq 1$.
- **Reflections:** We have already commented that $z \mapsto \overline{z}$ denotes reflection in the real axis.

  More generally, any reflection about the origin has the form $z \mapsto a\overline{z} + b$ where $a, b \in \mathbb{C}$ and $|a| = 1$.

What we have listed here are three types of isometry of $\mathbb{C}$. An **isometry** of $\mathbb{C}$ is a map $f : \mathbb{C} \to \mathbb{C}$ which preserves distance — that is for any two points z and w in $\mathbb{C}$ the distance between $f(z)$ and $f(w)$ equals the distance between a and b. Mathematically this means

$$|f(z) - f(w)| = |z - w|$$

for any complex numbers z and w. The following theorem, the proof of which is omitted here, characterises the isometries of $\mathbb{C}$.

Theorem 18 *Let $f : \mathbb{C} \to \mathbb{C}$ be an isometry. Then there exist complex numbers a and b with $|a| = 1$ such that*

$$f(z) = az + b \quad \text{or} \quad f(z) = a\overline{z} + b$$

for each $z \in \mathbb{C}$.

Example 19 *Express in the form $f(z) = a\overline{z} + b$ reflection in the line $x + y = 1$.*

Solution. Method One: Knowing from the previous theorem that the reflection has the form $f(z) = a\overline{z} + b$ we can find a and b by considering where two points go to. As 1 and i both lie on the line of reflection then they are both fixed. So

$$\begin{aligned}
a1 + b &= a\overline{1} + b = 1, \\
-ai + b &= a\overline{i} + b = i.
\end{aligned}$$

Substituting $b = 1 - a$ into the second equation we find

$$a = \frac{1-i}{1+i} = -i,$$

and $b = 1 + i$. Hence

$$f(z) = -i\overline{z} + 1 + i.$$

Method Two: We introduce as alternative method here — the idea of changing co-ordinates. We take a second set of complex co-ordinates in which the point $z = 1$ is the origin and for which the line of reflection is the real axis. The second complex co-ordinate w is related to the first co-ordinate z by

$$w = (1+i)(z-1).$$

For example, when $z = 1$ then $w = 0$, when $z = i$ then $w = -2$, when $z = 2 - i$ then $w = 2$, when $z = 2 + i$ then $w = 2i$. The real axis for the w co-ordinate has equation $x + y = 1$ and the imaginary axis has equation $y = x - 1$ in terms of our original co-ordinates.

The point to all this is that as w's real axis is the line of reflection then the transformation we're interested in is given by $w \mapsto \overline{w}$ in the new co-ordinates. Take then a point with complex co-ordinate z in our original co-ordinates system. Its w-co-ordinate is $(1+i)(z-1)$ — note we haven't moved the point yet, we've just changed what co-ordinates we're using. Now if we reflect the point we know the w-co-ordinate of the new point is $\overline{(1+i)(z-1)} = (1-i)(\overline{z}-1)$. Finally to get from the w-co-ordinate of the image point to the z-co-ordinate we reverse the co-ordinate change to get

$$\frac{(1-i)(\overline{z}-1)}{1+i} + 1 = -i(\overline{z}-1) + 1 = -i\overline{z} + i + 1$$

as required. ■

2 Their Analysis

2.1 The Complex Exponential Function

The real exponential function e^x (or $\exp x$) can be defined in several different ways. One such definition is by power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

The above infinite sum converges for all real values of x. What this means, is that for any real value of our *input* x, as we add more and more of the terms from the infinite sum above we generate a list of numbers (called *partial sums*) which get closer and closer to some value — this value we denote e^x. Different inputs will mean the sum converges to different answers. As an example, let's consider the case when $x = 2$:

1 term:	1	$= 1.0000$	6 terms:	$1 + \cdots + \frac{32}{120}$	$\cong 7.2667$
2 terms:	$1 + 2$	$= 3.0000$	7 terms	$1 + \cdots + \frac{64}{720}$	$\cong 7.3556$
3 terms:	$1 + 2 + \frac{4}{2}$	$= 5.0000$	8 terms	$1 + \cdots + \frac{128}{5040}$	$\cong 7.3810$
4 terms:	$1 + \cdots + \frac{8}{6}$	$\cong 6.3333$	9 terms	$1 + \cdots + \frac{256}{40320}$	$\cong 7.3873$
5 terms:	$1 + \cdots + \frac{16}{24}$	$= 7.0000$	∞ terms	e^2	$\cong 7.3891$

This idea of a power series defining a function should not be too alien — it is likely that you have already seen that the infinite geometric progression

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

converges to $(1-x)^{-1}$, at least when $|x| < 1$. This is another example of a power series defining a function.

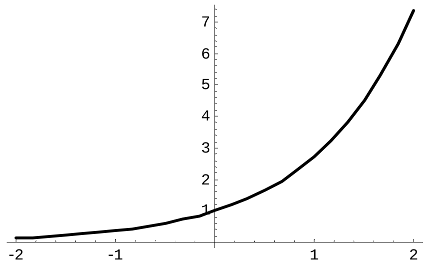
Proposition 20 *Let x be a real number. Then*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

converges to a real value which we shall denote as e^x. The function e^x has the following properties

$$\begin{array}{ll} (i) & \dfrac{d}{dx}e^x = e^x, \qquad e^0 = 1; \\ (ii) & e^{x+y} = e^x e^y \quad \textit{for any real } x, y; \\ (iii) & e^x > 0 \textit{ for any real } x. \end{array}$$

and a sketch of the exponential's graph is given below.



The graph of $y = e^x$.

Note that when $x = 1$ *this gives us the identity*

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots \cong 2.718.$$

We will simply take as read that of e^x — these are facts proven during the first term's analysis course. We can use either the power series definition, or one equivalent to property (i), to define the **complex exponential function**.

Proposition 21 *Let* z *be a complex number. Then*

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

converges to a complex value which we shall denote as e^z. *The function* e^z *has the following properties*

$$\begin{array}{rl} (i) & \dfrac{d}{dz}e^z = e^z, \qquad e^0 = 1; \\ (ii) & e^{z+w} = e^z e^w \ \text{ for any complex } z, w; \\ (iii) & e^z \neq 0 \ \text{ for any complex } z. \end{array}$$

By taking more and more terms in the series, we can calculate e^z to greater and greater degrees of accuracy as before. For example, to calculate e^{1+i} we see

1 term:	1	$= 1.0000$
2 terms:	$1 + (1+i)$	$= 2.0000 + 1.0000i$
3 terms:	$1 + (1+i) + \frac{2i}{2}$	$= 2.0000 + 2.0000i$
4 terms:	$1 + \cdots + \frac{-2+2i}{6}$	$\cong 1.6667 + 2.3333i$
5 terms:	$1 + \cdots + \frac{-4}{24}$	$\cong 1.5000 + 2.3333i$
6 terms:	$1 + \cdots + \frac{-4-4i}{120}$	$\cong 1.4667 + 2.3000i$
7 terms:	$1 + \cdots + \frac{-8i}{720}$	$\cong 1.4667 + 2.2889i$
8 terms:	$1 + \cdots + \frac{8-8i}{5040}$	$\cong 1.4683 + 2.2873i$
9 terms:	$1 + \cdots + \frac{16}{40320}$	$\cong 1.4687 + 2.2873i$
∞ terms:	e^{1+i}	$\cong 1.4687 + 2.2874i$

Their are two other important functions, known as hyperbolic functions, which are closely related to the exponential function — namely **hyperbolic cosine** $\cosh z$ and **hyperbolic sine** $\sinh z$.

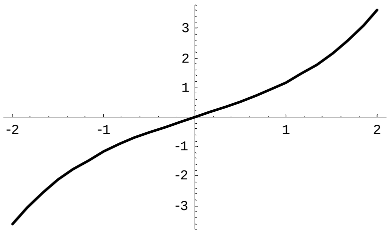
Definition 22 *Let* z *be a complex number. Then we define*

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

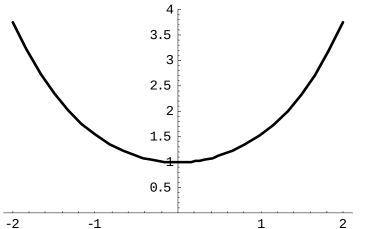
Corollary 23 *Hyperbolic sine and hyperbolic cosine have the following properties (which can easily be derived from the properties of the exponential function given in Proposition 21). For complex numbers* z *and* w*:*

$$\begin{array}{rl} (i) & \cosh z = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \cdots + \dfrac{z^{2n}}{(2n)!} + \cdots \\ (ii) & \sinh z = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots + \dfrac{z^{2n+1}}{(2n+1)!} + \cdots \\ (iii) & \dfrac{d}{dz}\cosh z = \sinh z \ \text{ and } \ \dfrac{d}{dz}\sinh z = \cosh z, \\ (iv) & \cosh(z+w) = \cosh z \cosh w + \sinh z \sinh w, \\ (v) & \sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w, \\ (vi) & \cosh(-z) = \cosh z \ \text{ and } \ \sinh(-z) = -\sinh z. \end{array}$$

and graphs of the sinh *and* cosh *are sketched below for real values of* x



The graph of $y = \sinh x$



The graph of $y = \cosh x$

2.2 The Complex Trigonometric Functions

The real functions **sine** and **cosine** can similarly be defined by power series and other characterising properties. Note that these definitions give us the sine and cosine of x *radians.*

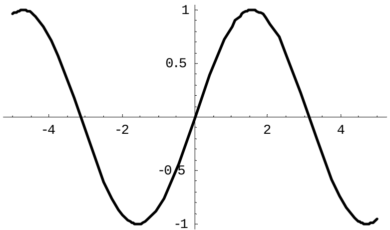
Proposition 24 *Let* x *be a real number. Then*

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots, \quad \textit{and}$$
$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots$$

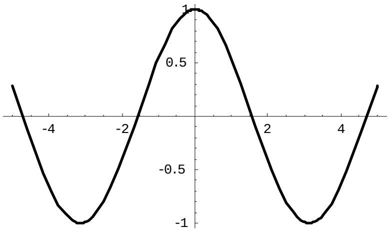
converge to real values which we shall denote as $\cos x$ *and* $\sin x$*. The functions* $\cos x$ *and* $\sin x$ *have the following properties*

$$\begin{array}{rl}
\textit{(i)} & \dfrac{d^2}{dx^2}\cos x = -\cos x, \qquad \cos 0 = 1, \quad \cos' 0 = 0, \\
\textit{(ii)} & \dfrac{d^2}{dx^2}\sin x = -\sin x, \qquad \sin 0 = 0, \quad \sin' 0 = 1, \\
\textit{(iii)} & \dfrac{d}{dx}\cos x = -\sin x, \quad \textit{and} \quad \dfrac{d}{dx}\sin x = \cos x, \\
\textit{(iv)} & -1 \le \cos x \le 1 \quad \textit{and} \quad -1 \le \sin x \le 1, \\
\textit{(v)} & \cos(-x) = \cos x \quad \textit{and} \quad \sin(-x) = -\sin x.
\end{array}$$

Property (i) above characterises $\cos x$ *and property (ii) characterises* $\sin x$ *— that is* $\cos x$ *and* $\sin x$ *are the unique real functions with these respective properties.*



The graph of $y = \sin x$



The graph of $y = \cos x$

As before we can extend these power series to the complex numbers to define the complex trigonometric functions.

Proposition 25 *Let* z *be a complex number. Then the series*

$$1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots+(-1)^n\frac{z^{2n}}{(2n)!}+\cdots, \quad \textit{and}$$
$$z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)!}+\cdots$$

converge to complex values which we shall denote as $\cos z$ *and* $\sin z$*. The functions* $\cos$ *and* $\sin$ *have the following properties*

$$\begin{array}{rl}
\textit{(i)} & \dfrac{d^2}{dz^2}\cos z = -\cos z, \qquad \cos 0 = 1, \quad \cos' 0 = 0, \\
\textit{(ii)} & \dfrac{d^2}{dz^2}\sin z = -\sin z, \qquad \sin 0 = 0, \quad \sin' 0 = 1, \\
\textit{(iii)} & \dfrac{d}{dz}\cos z = -\sin z, \quad \textit{and} \quad \dfrac{d}{dz}\sin z = \cos z, \\
\textit{(iv)} & \textit{Neither } \sin \textit{ nor } \cos \textit{ is bounded on the complex plane,} \\
\textit{(v)} & \cos(-z) = \cos z \quad \textit{and} \quad \sin(-z) = -\sin z.
\end{array}$$

Example 26 *Prove that* $\cos^2 z + \sin^2 z = 1$ *for all complex numbers z. (Note that, as we are dealing with complex numbers, this does not imply that* $\cos z$ *and* $\sin z$ *have modulus less than or equal to 1.)*

Solution. Define

$$F(z) = \sin^2 z + \cos^2 z.$$

Differentiating F, using the previous proposition and the product rule we see

$$F'(z) = 2\sin z\cos z + 2\cos z \times (-\sin z) = 0.$$

As the derivative $F' = 0$ then F must be constant. We note that

$$F(0) = \sin^2 0 + \cos^2 0 = 0^2 + 1^2 = 1$$

and hence $F(z) = 1$ for all z. ■

Contrast this with:

Example 27 *Prove that* $\cosh^2 - \sinh^2 z = 1$ *for all complex numbers z.*

Solution. We could argue similarly to the above. Alternatively as

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

and using $e^z e^{-z} = e^{z-z} = e^0 = 1$ from Proposition 21 we see

$$\begin{aligned}
\cosh^2 z - \sinh^2 z &= \left[\frac{(e^z)^2 + 2e^z e^{-z} + (e^z)^2}{4}\right] - \left[\frac{(e^z)^2 - 2e^z e^{-z} + (e^{-z})^2}{4}\right] \\
&= \frac{4e^z e^{-z}}{4} = 1.
\end{aligned}$$

■

It is for these reasons that the functions cosh and sinh are called hyperbolic functions and the functions sin and cos are often referred to as the circular functions. From the first example above we see that the point $(\cos t, \sin t)$ lies on the circle $x^2 + y^2 = 1$. As we vary t between 0 and 2π this point moves once anti-clockwise around the unit circle. In contrast, the point $(\cosh t, \sinh t)$ lies on the curve $x^2 - y^2 = 1$. This is the equation of a *hyperbola.* As t varies through the reals then $(\cosh t, \sinh t)$ maps out all of the right branch of the hyperbola. We can obtain the left branch by varying the point $(-\cosh t, \sinh t)$.

2.3 Identities

From looking at the graphs of $\exp x, \sin x, \cos x$ for real values of x it seems unlikely that all three functions can be related. The $\sin x$ and $\cos x$ are just out-of-phase but the exponential is unbounded unlike the trigonometric functions and has no periodicity. However, once viewed as functions of a complex variable, it is relatively easy to demonstrate a fundamental identity connecting the three. The following is due to Euler, dating from 1740.

Theorem 28 *Let z be a complex number. Then*

$$e^{iz} = \cos z + i\sin z.$$

Proof. Firstly note that the sequence i^n of powers of i goes $1, i, -1, -i, 1, i, -1, -i, 1, \ldots$ repeating forever with period 4. So, recalling the power series definitions of the exponential and trigonometric functions from Propositions 21 and 25, we see

$$\begin{aligned} e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \cdots \\ &= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} + \cdots \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &= \cos z + i \sin z. \end{aligned}$$

■

- Note that $\cos z \neq \operatorname{Re} e^{iz}$ and $\sin z \neq \operatorname{Im} e^{iz}$ in general for complex z.
- When we put $z = \pi$ into this proposition we find

  $$e^{i\pi} = -1.$$

  This is sometimes referred to as **Euler's Equation**, and is often credited as being the most beautiful equation in all of mathematics because it relates the fundamental constants $1, i, \pi, e$.
- Note that the complex exponential function has period $2\pi i$. That is

  $$e^{z+2\pi i} = e^z \text{ for all complex numbers } z.$$

- More generally when θ is a real number we see that

  $$e^{i\theta} = \cos\theta + i\sin\theta$$

  and so the polar form of a complex number from equation (3) is often written as

  $$z = re^{i\theta}.$$

  Moreover in these terms, De Moivre's Theorem (see Theorem 12) is the less surprising identity

  $$\left(e^{i\theta}\right)^n = e^{i(n\theta)}.$$

- If $z = x + iy$ then

  $$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

  and so

  $$|e^z| = e^x \quad \text{and} \quad \arg e^z = y.$$

As a corollary to the previous theorem we can now express $\cos z$ and $\sin z$ in terms of the exponential. We note

Corollary 29 *Let z be a complex number. Then*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \textit{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and

$$\begin{aligned} \cosh z &= \cos iz \quad \textit{and} \quad i\sinh z = \sin iz \\ \cos z &= \cosh iz \quad \textit{and} \quad i\sin z = \sinh iz. \end{aligned}$$

Proof. As cos is even and sin is odd then

$$e^{iz} = \cos z + i\sin z \quad \text{and} \quad e^{-iz} = \cos z - i\sin z.$$

Solving for $\cos z$ and $\sin z$ from these simultaneous equations we arrive at the required expressions. The others are easily verified from our these new expressions for cos and sin and our previous ones for cosh and sinh. ■

2.4 Applications

We can now turn these formula towards some applications and calculations. The following demonstrates, for one specific case, how formulae for $\cos nz$ and $\sin nz$ can be found in terms of powers of $\sin z$ and $\cos z$. The second example demonstrates a specific case of the reverse process — writing powers of $\cos z$ or $\sin z$ as combinations of $\cos nz$ and $\sin nz$ for various n.

Example 30 *Show that*

$$\cos 5z = 16\cos^5 z - 20\cos^3 z + 5\cos z.$$

Solution. Recall from De Moivre's Theorem that

$$(\cos z + i\sin z)^5 = \cos 5z + i\sin 5z.$$

Now if x and y are real then by the Binomial Theorem

$$(x+iy)^5 = x^5 + 5ix^4y - 10x^3y^2 - 10ix^2y^3 + 5xy^4 + iy^5.$$

Hence

$$\begin{aligned}\cos 5\theta &= \operatorname{Re}(\cos\theta + i\sin\theta)^5\\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)^2\\ &= (1+10+5)\cos^5\theta + (-10-10)\cos^3\theta + 5\cos\theta\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.\end{aligned}$$

This formula in fact holds true when θ is a general complex number and not just for reals. ■

Example 31 *Let z be a complex number. Prove that*

$$\sin^4 z = \frac{1}{8}\cos 4z - \frac{1}{2}\cos 2z + \frac{3}{8}.$$

Hence find the power series for $\sin^4 z$.

Solution. We have that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

So

$$\begin{aligned}\sin^4 z &= \frac{1}{(2i)^4}\left(e^{iz} - e^{-iz}\right)^4\\ &= \frac{1}{16}\left(e^{4iz} - 4e^{2iz} + 6 - 4e^{-2iz} + e^{-4iz}\right)\\ &= \frac{1}{16}\left(\left(e^{4iz} + e^{-4iz}\right) - 4\left(e^{2iz} + e^{-2iz}\right) + 6\right)\\ &= \frac{1}{16}(2\cos 4z - 8\cos 2z + 6)\\ &= \frac{1}{8}\cos 4z - \frac{1}{2}\cos 2z + \frac{3}{8},\end{aligned}$$

as required. Now $\sin^4 z$ has only even powers of z^{2n} in its power series. From our earlier power series for $\cos z$ we see, when $n > 0$, the coefficient of z^{2n} will equals

$$\frac{1}{8}\times(-1)^n\frac{4^{2n}}{(2n)!} - \frac{1}{2}\times(-1)^n\frac{2^{2n}}{(2n)!} = (-1)^n\frac{2^{4n-3} - 2^{2n-1}}{(2n)!}z^{2n}$$

which we note is zero when $n = 1$. Also when $n = 0$ we see that the constant term is $1/8 - 1/2 + 3/8 = 0$. So the required power series is

$$\sin^4 z = \sum_{n=2}^{\infty}(-1)^n\frac{2^{4n-3} - 2^{2n-1}}{(2n)!}z^{2n}.$$

■

Example 32 *Prove for any complex numbers z and w that*

$$\sin(z+w) = \sin z \cos w + \cos z \sin w.$$

Solution. Recalling the expressions for sin and cos from Corollary 29 we have ■

$$\begin{aligned}\text{RHS} &= \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right)+\left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right)\\ &= \frac{2e^{iz}e^{iw}-2e^{-iz}e^{-iw}}{4i}\\ &= \frac{e^{i(z+w)}-e^{-i(z+w)}}{2i} = \sin(z+w) = \text{LHS}.\end{aligned}$$

Example 33 *Prove that for complex z and w*

$$\sin(z+iw) = \sin z \cosh w + i \cos z \sinh w.$$

Solution. Use the previous example, recalling that $\cos(iw) = \cosh w$ and $\sin(iw) = i \sinh w$. ■

Example 34 *Let x be a real number and n a natural number. Show that*

$$\sum_{k=0}^{n} \cos kx = \frac{\cos \frac{n}{2}x \sin \frac{n+1}{2}x}{\sin \frac{1}{2}x} \quad \textit{and} \quad \sum_{k=0}^{n} \sin kx = \frac{\sin \frac{n}{2}x \sin \frac{n+1}{2}x}{\sin \frac{1}{2}x}$$

Solution. As $\cos kx + i \sin kx = \left(e^{ix}\right)^k$ then these sums are the real and imaginary parts of a geometric series, with first term 1, common ration e^{ix} and $n+1$ terms in total. So recalling

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1},$$

we have

$$\begin{aligned}\sum_{k=0}^{n} \left(e^{ix}\right)^k &= \frac{e^{(n+1)ix}-1}{e^{ix}-1}\\ &= \frac{e^{inx/2}\left(e^{(n+1)ix/2}-e^{-(n+1)ix/2}\right)}{e^{ix/2}-e^{-ix/2}}\\ &= e^{inx/2}\frac{2i \sin \frac{n+1}{2}x}{2i \sin \frac{1}{2}x}\\ &= \left(\cos \frac{nx}{2} + i \sin \frac{nx}{2}\right)\frac{\sin \frac{n+1}{2}x}{\sin \frac{1}{2}x}.\end{aligned}$$

The results follow by taking real and imaginary parts. Again this identity holds for complex values of x as well. ■

3 Exercises

Basic Algebra:

Exercise 1 *Put each of the following numbers into the form $a+bi$.*

$$(1+2i)(3-i), \quad \frac{1+2i}{3-i}, \quad (1+i)^4.$$

Exercise 2 *Let $z_1 = 1+i$ and let $z_2 = 2-3i$. Put each of the following into the form $a+bi$.*

$$z_1 + z_2, \quad z_1 - z_2, \quad z_1 z_2, \quad z_1/z_2, \quad \bar{z}_1 \bar{z}_2.$$

Exercise 3 *Find the modulus and argument of each of the following numbers.*

$$1+\sqrt{3}i, \quad (2+i)(3-i), \quad (1+i)^5, \quad \frac{(1+2i)^3}{(2-i)^3}.$$

Exercise 4 *Let* α *be a real number in the range* $0 < \alpha < \pi/2$. *Find the modulus and argument of the following numbers.*

$$\cos\alpha - i\sin\alpha, \quad \sin\alpha - i\cos\alpha, \quad 1 + i\tan\alpha, \quad 1 + \cos\alpha + i\sin\alpha.$$

Exercise 5 *Let* z *and* w *be two complex numbers such that* $zw = 0$. *Show either* $z = 0$ *or* $w = 0$.

Exercise 6 *Prove that every non-zero complex number has two square roots.*

Exercise 7 *Prove the remaining identities from Proposition 11.*

Polynomial Equations:

Exercise 8 *Find the square roots of* $-5 - 12i$, *and hence solve the quadratic equation*

$$z^2 - (4 + i)\,z + (5 + 5i) = 0.$$

Exercise 9 *Show that the complex number* $1 + i$ *is a root of the cubic equation*

$$z^3 + z^2 + (5 - 7i)\,z - (10 + 2i) = 0,$$

and hence find the other two roots.

Exercise 10 *Show that the complex number* $2 + 3i$ *is a root of the quartic equation*

$$z^4 - 4z^3 + 17z^2 - 16z + 52 = 0,$$

and hence find the other three roots.

Exercise 11 *Let* p *and* q *be real numbers with* $p \leq 0$. *Find the co-ordinates of the turning points of the cubic* $y = x^3 + px + q$. *Show that the cubic equation* $x^3 + px + q = 0$ *has three real roots, with two or more repeated, precisely when*

$$4p^3 + 27q^2 = 0.$$

Under what conditions on p *and* q *does* $x^3 + px + q = 0$ *have (i) three distinct real roots, (ii) just one real root? How many real roots does the equation* $x^3 + px + q = 0$ *have when* $p > 0$*?*

Exercise 12 *Consider the cubic equation* $z^3 + mz + n = 0$ *where* m *and* n *are real numbers. Let* Δ *be a square root of* $(n/2)^2 + (m/3)^3$. *We then define* t *and* u *by*

$$t = -n/2 + \Delta \quad \textit{and} \quad u = n/2 + \Delta,$$

and let T *and* U *respectively be cube roots of* t *and* u. *Show that* tu *is real, and that if* T *and* U *are chosen appropriately, then* $z = T - U$ *is a solution of the original cubic equation.*

Use this method to completely solve the equation $z^3 + 6z = 20$. *By making a substitution of the form* $w = z - a$ *for a suitable choice of* a, *find all three roots of the equation* $8w^3 + 12w^2 + 54w = 135$.

De Moivre's Theorem and Roots of Unity

Exercise 13 *Use De Moivre's Theorem to show that*

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1, \quad \textit{and} \quad \sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

Exercise 14 *Show that*

$$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$$

and hence find $\int_0^{\pi/2} \cos^5\theta \; d\theta$.

Exercise 15 *Let*

$$\zeta = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}.$$

Show that $\zeta^5 = 1$, *and deduce that*

$$1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0.$$

Find the quadratic equation with roots $\zeta + \zeta^4$ *and* $\zeta^2 + \zeta^3$. *Hence show that*

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}.$$

Exercise 16 *Determine the modulus and argument of the two complex numbers* $1+i$ *and* $\sqrt{3}+i$*. Also write the number* $(1+i)\,/\,(\sqrt{3}+i)$ *in the form* $x+iy$*. Deduce that*

$$\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}\quad\text{and}\quad\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Exercise 17 *By considering the seventh roots of* -1 *show that*

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}.$$

What is the value of

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\,?$$

Exercise 18 *Find all the roots of the equation* $z^8=-1$*. Hence, write* z^8+1 *as the product of four quadratic factors.*

Exercise 19 *Find all the roots of the following equations.*

1. $1+z^2+z^4+z^6=0$,
2. $1+z^3+z^6=0$,
3. $(1+z)^5-z^5=0$,
4. $(z+1)^9+(z-1)^9=0$.

Exercise 20 *Express* $\tan 7\theta$ *in terms of* $\tan\theta$ *and its powers. Hence solve the equation*

$$x^6-21x^4+35x^2-7=0.$$

Geometry and the Argand Diagram

Exercise 21 *On separate Argand diagrams sketch the following sets:*

1. $|z|<1$;
2. $\operatorname{Re} z=3$;
3. $|z-1|=|z+i|$;
4. $-\pi/4<\arg z<\pi/4$;
5. $\operatorname{Re}(z+1)=|z-1|$;
6. $\arg(z-i)=\pi/2$;
7. $|z-3-4i|=5$;
8. $\operatorname{Re}((1+i)\,z)=1$.
9. $\operatorname{Im}(z^3)>0$.

Exercise 22 *Multiplication by* i *takes the point* $x+iy$ *to the point* $-y+ix$*. What transformation of the Argand diagram does this represent? What is the effect of multiplying a complex number by* $(1+i)\,/\sqrt{2}$*? [Hint: recall that this is square root of* i*.]*

Exercise 23 *Let* ABC *be a triangle in* $\mathbb{C}$ *with vertices* $A=1+i, B=2+3i, C=5+2i$*. Write, in the form* $a+bi$*, the images of the three vertices when the plane is rotated about* 0 *through* $\pi/3$ *radians anti-clockwise.*

Exercise 24 *Let* $a,b\in\mathbb{C}$ *with* $|a|=1$*. Show directly that the map* $f:\mathbb{C}\to\mathbb{C}$ *given by* $f(z)=az+b$ *preserves distances and angles.*

Exercise 25 *Write in the form* $z\mapsto az+b$ *the rotation through* $\pi/3$ *radians anti-clockwise about the point* $2+i$*.*

Exercise 26 *Write in the form* $z\mapsto a\bar{z}+b$ *the reflection in the line* $3x+2y=6$*.*

Exercise 27 *Let* $f:\mathbb{C}\to\mathbb{C}$ *be given by* $f(z)=iz+3-i$. *Find a map* $g:\mathbb{C}\to\mathbb{C}$ *of the form* $g(z)=az+b$ *where* $|a|=1$ *such that*

$$g(g(z))=f(z)\,.$$

How many such maps g *are there? Geometrically what transformations do these maps* g *and the map* f *represent?*

Exercise 28 *Find two reflections* $h:\mathbb{C}\to\mathbb{C}$ *and* $k:\mathbb{C}\to\mathbb{C}$ *such that* $k(h(z))=iz$ *for all* z.

Exercise 29 *What is the centre of rotation of the map* $z\mapsto az+b$ *where* $|a|=1, a\neq 1$*? What is the invariant line of the reflection* $z\mapsto a\bar{z}+b$ *where* $|a|=1$*?*

Exercise 30 *Let* t *be a real number. Find expressions for*

$$x=\operatorname{Re}\frac{1}{2+ti},\qquad y=\operatorname{Im}\frac{1}{2+ti}.$$

Find an equation relating x *and* y *by eliminating* t. *Deduce that the image of the line* $\operatorname{Re}z=2$ *under the map* $z\mapsto 1/z$ *is contained in a circle. Is the image of the line all of the circle?*

Exercise 31 *Find the image of the line* $\operatorname{Re}z=2$ *under the maps*

$$z\mapsto iz,\qquad z\mapsto z^2,\qquad z\mapsto e^z,\qquad z\mapsto\sin z,\qquad z\mapsto\frac{1}{z-1}.$$

Exercise 32 *Draw the following parametrised curves in* $\mathbb{C}$.

$$\begin{aligned}
z(t) &= e^{it}, \qquad (0\leqslant t\leqslant\pi)\,;\\
z(t) &= 3+4i+5e^{it}, \qquad (0\leqslant t\leqslant 2\pi)\,;\\
z(t) &= t+i\cosh t, \qquad (-1\leqslant t\leqslant 1)\,;\\
z(t) &= \cosh t+i\sinh t, \qquad (t\in\mathbb{R})\,;
\end{aligned}$$

Exercise 33 *Prove, using complex numbers, that the midpoints of the sides of an arbitrary quadrilateral are the vertices of a parallelogram.*

Exercise 34 *Let* z_1 *and* z_2 *be two complex numbers. Show that*

$$|z_1-z_2|^2+|z_1+z_2|^2=2\left(|z_1|^2+|z_2|^2\right).$$

This fact is called the Parallelogram Law — how does this relate the lengths of the diagonals and sides of the parallelogram? [Hint: consider the parallelogram in $\mathbb{C}$ *with vertices* $0,z_1,z_2,z_1+z_2$*.]*

Exercise 35 *Consider a quadrilateral* $OABC$ *in the complex plane whose vertices are at the complex numbers* $0,a,b,c$. *Show that the equation*

$$|b|^2+|a-c|^2=|a|^2+|c|^2+|a-b|^2+|b-c|^2$$

can be rearranged as

$$|b-a-c|^2=0.$$

Hence show that the only quadrilaterals to satisfy the Parallelogram Law are parallelograms.

Exercise 36 *Let* $\omega=e^{2\pi i/3}=\left(-1+\sqrt{3}i\right)/2$. *Show that a triangle* ABC*, where the vertices are read anti-clockwise, is equilateral if and only if*

$$A+\omega B+\omega^2 C=0.$$

Exercise 37 *(Napoleon's Theorem) Let* ABC *be an arbitrary triangle. Place three equilateral triangles* ABD, BCE, CAF*, one on each face and pointing outwards. Show that the centroids of these three new triangles define a fourth equilateral triangle. [The centroid of a triangle whose vertices are represented by the complex numbers* a,b,c *is the point represented by* $(a+b+c)/3$*.]*

Exercise 38 *Let* A,C *be real numbers and* B *be a complex number. Consider the equation*

$$Az\bar{z}+\bar{B}z+B\bar{z}+C=0. \tag{17}$$

Show that if $A=0$*, then equation (17) defines a line. Conversely show that any line can be put in this form with* $A=0$.

Show that if $A\neq 0$ *then equation (17) defines a circle, a single point or has no solutions. Under what conditions on* A,B,C *do the solutions form a circle and, assuming the condition holds, determine the radius and centre of the circle.*

Exercise 39 *Determine the equation of the following circles and lines in the form*

$$Az\bar{z} + \bar{B}z + B\bar{z} + C = 0 \quad \textit{where } B \in \mathbb{C} \textit{ and } A, C \in \mathbb{R}.$$

1. *The circle with centre* $3 + 4i$ *and radius* 5.
2. *The circle which passes through* $1, 3$ *and* i
3. *The line through* $1 + 3i$ *and* $2 - i$.
4. *The line through* 2 *and making an angle* θ *with the real-axis.*

Exercise 40 *Find the image under the map* $z \mapsto 1/z$ *of the two circles and two lines in the previous exercise. Ensure that your answers are all in the same form as the equation (17).*

Analysis and Power Series

Exercise 41 *Find the real and imaginary parts, and the magnitude and argument of the following.*

$$e^{3+2i}, \quad \sin(4+2i), \quad \cosh(2-i), \quad \tanh(1+2i).$$

Exercise 42 *Find all the solutions of the following equations.*

$$\begin{aligned} e^z &= 1; \\ \cosh z &= -2; \\ \sin z &= 3. \end{aligned}$$

Exercise 43 *Show that*

$$\overline{\cos z} = \cos \bar{z}, \quad \textit{and} \quad \overline{\sin z} = \sin \bar{z}.$$

Show further that, if $z = x + iy$, *then*

$$|\sin z|^2 = \frac{1}{2}(\cosh 2y - \cos 2x); \qquad |\cos z|^2 = \frac{1}{2}(\cosh 2y + \cos 2x).$$

Sketch the regions $|\sin z| \leqslant 1$ *and* $|\cos z| \leqslant 1$.

Exercise 44 *Use the identities of the previous exercise to show that*

$$|\cos z|^2 + |\sin z|^2 = 1$$

if and only if z *is real.*

Exercise 45 *Show that*

$$\cosh^2 z = \frac{1}{2}(1 + \cosh 2z); \qquad \sinh^3 z = \frac{1}{4}(\sinh 3z - 3\sinh z).$$

Hence find the power series of $\cosh^2 z$ *and* $\sinh^3 z$.

Exercise 46 *Show that*

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y; \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

and

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

Exercise 47 *Let* $\omega = e^{2\pi i/3} = \left(-1 + \sqrt{3}\right)/2$ *and let* k *be an integer Show that*

$$1 + \omega^k + \omega^{2k} = 1 + 2\cos\frac{2\pi k}{3} = \begin{cases} 3 & \textit{if } k \textit{ is a multiple of 3;} \\ 0 & \textit{otherwise.} \end{cases}$$

Deduce that

$$\frac{1}{3}\left(e^z + e^{\omega z} + e^{w^2 z}\right) = \sum_{n=0}^{\infty} \frac{z^{3n}}{(3n)!}.$$

Determine

$$\sum_{n=0}^{\infty} \frac{8^n}{(3n)!}$$

ensuring that your answer is in a form that is evidently a real number.